Modeling piecewise linear functions

Def: function f is piece-wise linear if its domain can be partitioned into intervals in which f is linear

Today: Find $\bar{x}$ that maximizes pw-linear function $f(x)$.

Ex:
- Waxxon Oil produces two types of gasolines
  gas 1 and gas 2
  from crude oils oil 1 and oil 2
- Blending restrictions
  gas 1 $\geq$ 50% oil 1
  gas $\geq$ 60% oil 1
- Sales price p. gallon:
  gas 1 : 30 cents
  gas 2 : 28 cents
- Waxxon has 500 gallons of oil 1 and 1000 gallons of oil 2
  Can buy at most 1500 extra gallons of oil 1.
- Depending on quantity of oil 1 that they purchase, they receive volume discounts.

First 500 gallons	:	25 cents
next 500 gallons	:	20 cents
every gallon exc. 1000	:	15 cents

$\Longrightarrow$ price for x gallons of oil 1 :

$$c(x) = \begin{cases} 25x & : \; x \leq 500 \\ 20x + 2500 & : \; x \in (500, 1000] \\ 15x + 7500 & : \; x \in (1000, 1500] \end{cases}$$

$500 \cdot 25 + 20 \cdot (x - 500)$

$20 \cdot 1000 + 2500 + 15(x - 1000)$

Linear model:

x_{ij} : amount of oil i used in gas j

x : amount of extra gallons of oil 1 bought

$$\Longrightarrow \quad \max \quad 30 \cdot (x_{11} + x_{21}) + 38(x_{12} + x_{22}) - c(x)$$

s.t.

$$x_{12} + x_{12} \leq 500 + x \qquad \text{[oil 1 consump]}$$

$$x_{21} + x_{22} \leq 1000 \qquad \text{[oil 2 cons.]}$$

$$x_{11} \geq \tfrac{1}{2}(x_{11} + x_{21})$$

blending

$$x_{12} \geq .6\,(x_{12} + x_{22})$$

$$x_{ij} \geq 0 \;\; \forall ij, \quad 0 \leq x \leq 1500$$

Not linear!

Idea: express x as $\delta_1 + \delta_2 + \delta_3$ where

δ_i = # gallons bought at price p_i

$p_1 = 25$ cents , $p_2 = 20$ cents , $p_3 = 15$ cents

For any given $0 \le x \le 1500$ want

$$\delta_i = \max\{0, \min\{500, x - b_{i-1}\}\}$$

where $b_0 = 0$, $b_1 = 500$, $b_2 = 1000$ are the lower interval boundaries.

Ex: $x = 850$ $\quad \delta_1 = 500$, $\delta_2 = 350$, $\delta_3 = 0$

Add linear constraints

$$x = \delta_1 + \delta_2 + \delta_3$$
$$0 \le \delta_i \le 500$$

to LP and change objective to

$$\max \quad 30 \cdot (x_{11} + x_{21}) + 38 (x_{12} + x_{22}) - \left(25 \cdot \delta_1 + 20 \cdot \delta_2 + 15\, \delta_3\right)$$

What is wrong?

Suppose LP yields opt. soln with $500 < x < 1000$.

Can then choose

$$\delta_3 = 500 \quad \text{and} \quad \delta_2 = x - 500.$$

$\Longrightarrow$ LP pays too low a price.

Need to enforce use of lower indices first.

$$\delta_2 > 0 \quad \text{only if} \quad \delta_1 = 500$$
$$\delta_3 > 0 \quad \text{only if} \quad \delta_2 = 500$$

Do this via indicator variables u_1 and u_2:

$$x \ge 500 \quad \Longrightarrow \quad u_1 = 1$$
$$x \ge 1000 \quad \Longrightarrow \quad u_2 = 1$$

Then: $u_1 = 1 \Rightarrow \delta_1 = 500$ (1)

$u_2 = 1 \Rightarrow \delta_2 = 500$ (2)

$u_1 = 0 \Rightarrow \delta_2 = \delta_3 = 0$ (3)

$u_2 = 0 \Rightarrow \delta_3 = 0$ (4)

linear constraints

$$500 \cdot u_1 \overset{(1)}{\leq} \delta_1 \leq 500$$

$$500 \cdot u_2 \overset{(2)}{\leq} \delta_2 \overset{(3)*}{\leq} 500 \cdot u_1$$

$$0 \leq \delta_3 \overset{(4)}{\leq} 500 \cdot u_2$$

* This enforces $\delta_2 = 0$ if $u_1 = 0$ and also $u_2 = 0$.

$\Rightarrow \delta_3 \leq 500 \cdot u_2 = 0$.

New LP:

$$\max \quad 12 \cdot (x_{11} + x_{21}) + 14(x_{12} + x_{22}) - 25\delta_1 - 20\delta_2 - 15\delta_3$$

s.t.

$$x_{12} + x_{12} \leq 500 + x$$

$$x_{21} + x_{22} \leq 1000$$

$$x_{11} \geq \frac{1}{2}(x_{11} + x_{21})$$

$$x_{12} \geq .6\,(x_{12} + x_{22})$$

$$500\, u_1 \leq \delta_1 \leq 500$$

$$500\, u_2 \leq \delta_2 \leq 500\, u_1$$

$$0 \leq \delta_3 \leq 500\, u_2$$

$$x = \delta_1 + \delta_2 + \delta_3$$

$$x_{ij}, \delta \geq 0$$

$$u_1, u_2 \in \{0, 1\}$$

$$0 \leq x \leq 1500$$