Branch & Bound

Suppose we'd like to solve

$$\begin{array}{lll} \max & 3x_1 + 10x_2 & \text{(IP)} \\ \text{s.t.} & x_1 + 4x_2 \le 8 & (1) \\ & x_1 + x_2 \le 4 & (2) \\ & x_1, x_2 \ge 0 & (3) \\ & x_1, x_2 \text{ int} & (4) \end{array}$$

$$\mathcal{F} = \{ (x_1, x_2) \; : \; (1), (2), (3) \}$$

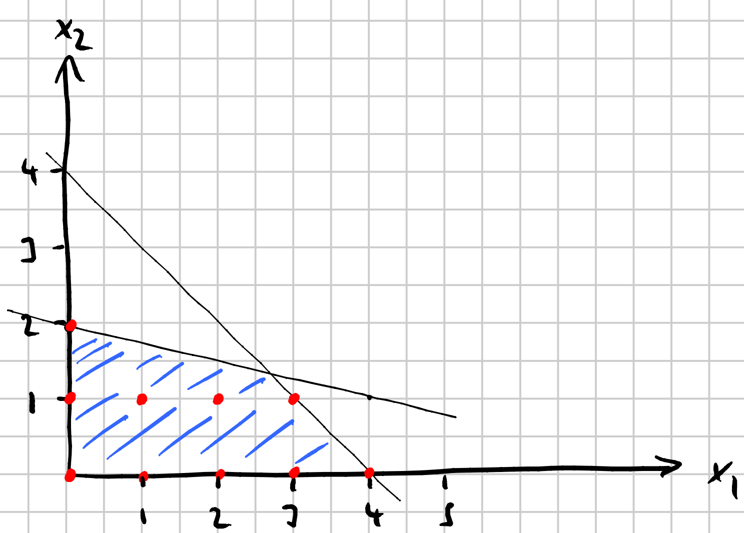
Ideas: 1) Must have $0 \le x_1 \le 4$, $0 \le x_2 \le 2$.

Enumerate all possible pairs in this range.

Approach does not scale.

2) Def: The LP obtained from (IP) by dropping (4) is called the LP relaxation of IP.

<u>Pictorial view:</u>



• feasible soln's for IP

feasible region for LP relax

Thm: Let (P) be the LP-relax of (IP) and let opt_P and opt_{IP} be their optimal values. Must have

$$opt_{IP} \le opt_P$$

Pf: Clearly any solution x that is feasible for (IP) is also feasible for (P). $\blacksquare$

Corollary: If (P) has an optimal integral solution x^*, then x^* is optimal for (IP) as well.

Idea: Solve LP (P) and hope....

Opt. solution is $x^* = (\frac{8}{3}, \frac{4}{3})$

with obj. value $\frac{64}{3}$.

Branch & Bound main idea: The LP solution tells us that

1) $opt_{IP} \le \lfloor \frac{64}{3} \rfloor = 21$

2) opt. solution of (IP) lies in

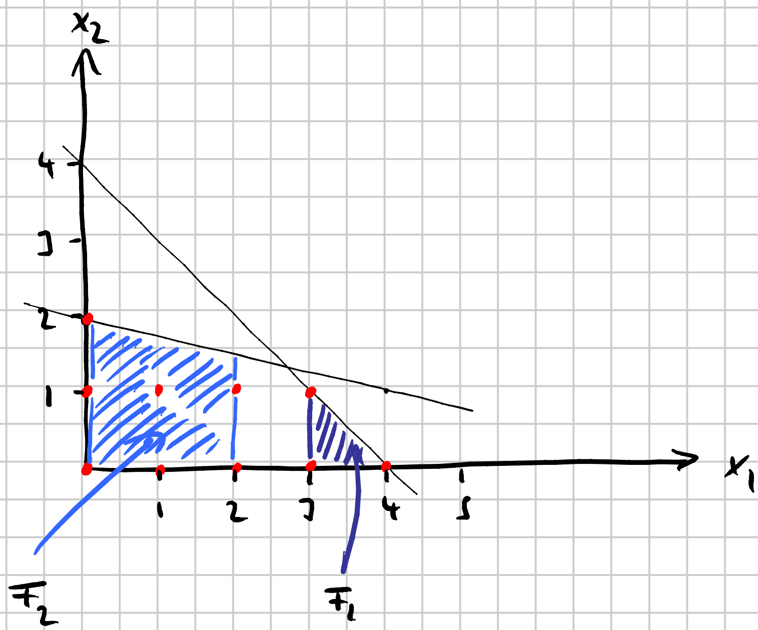
$$F_1 \cup F_2$$

where

$$F_2 = \{ (x_1, x_2) \in F : x_1 \le \lfloor \tfrac{8}{3} \rfloor \}$$

$$F_1 = \{ (x_1, x_2) \in F : x_2 \ge \lceil \tfrac{8}{3} \rceil \}$$

Pictorial view:



Solve

$$(P_1) \qquad \max \quad 3x_1 + 10x_2$$
$$\text{s.t.} \quad x \in F_1$$

$\Rightarrow \quad x_1^* = 3, \; x_2^* = 1 \qquad$ value 19

<u>We know:</u> ① $opt_{IP} \geq 19$

② $opt_{IP} \leq 21$

We're close but maybe there is a better IP solution in F_2.

Solve

$$(P_2) \qquad \max \quad 3x_1 + 10x_2$$
$$\text{s.t.} \quad x \in F_2$$

$\Rightarrow \quad x_1^* = 2, \; x_2^* = \frac{3}{2} \qquad$ opt value $opt_2 := 21$

What do we infer from this?

Best integer solution in F_2 has value at most 21.

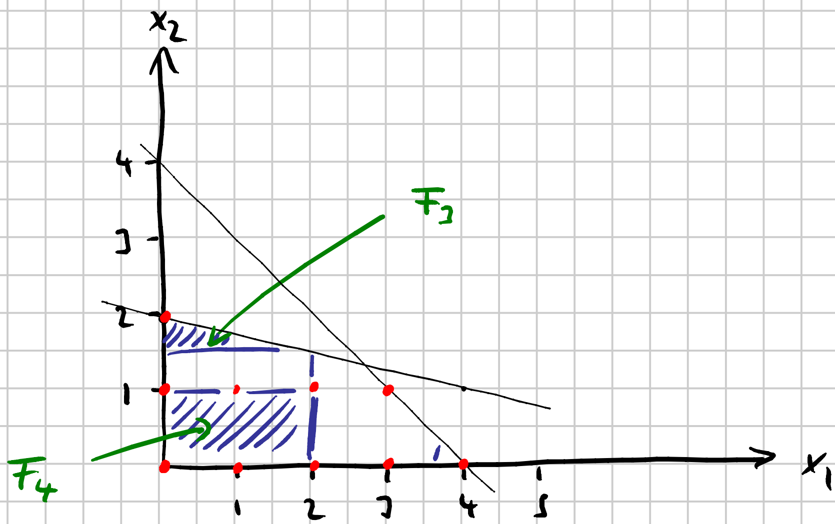
We have found an integer solution with value 19.

Potentially there is a better integer solution in F_2.

In fact, any integer solution in F_2 must be either in

$$F_3 = \{(x_1, x_2) \in F_2 : x_2 \geq \lceil 1.5 \rceil\}$$

or $$F_4 = \{(x_1, x_2) \in F_2 : x_2 \leq \lfloor 1.5 \rfloor\}$$



Solve

$$(P_3) \quad \max \quad 3x_1 + 10x_2$$
$$\text{s.t.} \quad x \in F_3$$

$\Longrightarrow$ $x_1^* = 0$, $x_2^* = 2$

$\text{opt}_3 := 20$

<u>Status:</u> 1) Have feasible IP solution with value 20

2) Know that best possible value of any solution in F_2 is $\text{opt}_2 = 21$.

$\Longrightarrow$ F_4 may have an IP soln of value 21

Solve (P_4) $\max$ $3x_1 + 10x_2$

s.t. $x \in F_4$

$\Longrightarrow$ $x_1^* = 2$, $x_2^* = 1$ $\quad \text{opt}_4 := 16$

Nothing left to explore! Opt. solution to (IP) is $(0,2)$ with val 20.

relax of IP — opt IP solution so far — its value

Branch & Bound $(P, \bar{x}, \bar{v})$

① Solve $P \longrightarrow (x^*, opt_P)$

② **if** $(opt_P \le \bar{v})$ or (P infeasible) **then**

   return $(\bar{x}, \bar{v})$;   (← Pruning)

③ **if** x^* integer **then** return (x^*, opt_P)

else

   Choose an arbitrary index i s.t. x_i^* fractional

Branch on x_i^*

   P_1 : add constr. $x_i \le \lfloor x_i^* \rfloor$ to P

   P_2 : add constr. $x_i \ge \lceil x_i^* \rceil$ to P

   $(x_1, v_1) \leftarrow B\&B(P_1, \bar{x}, \bar{v})$;

   if $(v_1 > \bar{v})$

      $\bar{v} := v_1$, $\bar{x} := x_1$

   end

   $(x_2, v_2) \leftarrow B\&B(P_2, \bar{x}, \bar{v})$

   if $(v_2 > \bar{v})$ **then**

      $\bar{v} := v_2$, $\bar{x} := x_2$

   end

   return $(\bar{x}, \bar{v})$

end

Implementation May have to solve large number of LP's!

Back to example. LP relax of IP is

$$\begin{aligned}\max \quad & 3x_1 + 10x_2 \\ \text{s.t.} \quad & x_1 + 4x_2 \le 8 \\ & x_1 + x_2 \le 4 \\ & x \ge 0\end{aligned}$$

Solve via Simplex.

Init tableau:

$$\begin{array}{c} z \\ s_1 \\ s_2 \end{array}\left[\begin{array}{cccc|c} -3 & -10 & & & 0 \\ \hline 1 & 4 & 1 & & 8 \\ 1 & 1 & & 1 & 4 \end{array}\right] \xrightarrow{\text{Simplex}} \underbrace{\begin{array}{c} z \\ x_1 \\ x_2 \end{array}\left[\begin{array}{cccc|c} & & 7/3 & 2/3 & 64/3 \\ \hline 1 & & -1/3 & 4/3 & 8/3 \\ & 1 & 1/3 & -1/3 & 4/3 \end{array}\right]}_{(*)}$$

<u>Recall:</u> $F_2 = \{ (x_1, x_2) \in F : x_1 \le \lfloor \frac{8}{3} \rfloor \}$

<u>Solve</u> $\max \; 3x_1 + 10x_2$

s.t. $x \in F_2$

Do we have to resolve from scratch or can we use (*)?

No. 1) Write new inequality in equality form

$$x_1 + s_3 = \lfloor \tfrac{8}{3} \rfloor$$

2) Add to tableau (*)

$$\begin{array}{c} z \\ x_1 \\ x_2 \\ s_2 \end{array}\left[\begin{array}{ccccc|c} & & 7/3 & 2/3 & & 64/3 \\ \hline 1 & & -1/3 & 4/3 & & 8/3 \\ & 1 & 1/3 & -1/3 & & 4/3 \\ 1 & & & & 1 & 2 \end{array}\right] \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array}$$

Not in standard form! Subtract (1) from (3).

$$
\begin{array}{c}
\left[\begin{array}{c|ccccc|c}
z & & & 7/3 & 2/3 & & 64/3 \\
\hline
x_1 & 1 & & -1/3 & 4/3 & & 8/3 \\
x_2 & & 1 & 1/3 & -1/3 & & 4/3 \\
s_2 & & & 1/3 & -4/3 & 1 & -2/3
\end{array}\right]
\begin{array}{c}
0 \\ 1 \\ 2 \\ 3
\end{array}
\\
\begin{array}{ccc}
\uparrow & \uparrow & \uparrow \\
s_1 & s_2 & s_3
\end{array}
\end{array}
$$

$\rightarrow$ Infeasible basis! But dual feasible (i.e., reduced costs of non-basic vars are non-positive).

3) Apply dual Simplex to new tableau.

<u>Leaving var:</u> s_2

Which var enters?

<u>Try:</u> 1) s_1

Have to multiply row (3) by 3.

$\Longrightarrow$ s_1 has val -2 in new basis. Bad!

2) s_2

Pivoting on (s_3, s_2) gives tableau:

$$
\left[\begin{array}{c|ccccc|c}
z & & & 5/2 & & 1/2 & 21 \\
\hline
x_1 & 1 & & & & 1 & 2 \\
x_2 & & 1 & 1/4 & & -1/4 & 3/2 \\
s_2 & & & -1/4 & 1 & -3/4 & 1/2
\end{array}\right]
$$

$\Longrightarrow$ optimal! $x_1 = 2$, $x_2 = 3/2$, $s_2 = 1/2$

Now in order to obtain P_3 we add $x_2 \geq \lceil 3/2 \rceil = 2$ to P_2.

Add to last tableau:

$$-x_2 + s_4 = -2$$

$$\begin{array}{c|cccccc|c|c}
z & & & 5/2 & & 1/2 & & 20 & 0 \\
\hline
x_1 & 1 & & & & 1 & & 2 & 1 \\
x_2 & & 1 & 1/4 & & -1/4 & & 3/2 & 2 \\
s_2 & & & -1/4 & 1 & -3/4 & & 1/2 & 3 \\
s_4 & & -1 & & & & 1 & -2 & 4
\end{array}$$

Add (2) to (4) :

$$\begin{array}{c|cccccc|c}
z & & & 5/2 & & 1/2 & & 20 \\
\hline
x_1 & 1 & & & & 1 & & 2 \\
x_2 & & 1 & 1/4 & & -1/4 & & 3/2 \\
s_2 & & & -1/4 & 1 & -3/4 & & 1/2 \\
s_4 & & & 1/4 & & -1/4 & 1 & -1/2
\end{array}$$

Use dual simplex again: s_4 leaves, s_3 enters

$$\begin{array}{c|cccccc|c}
z & & & 3 & & & 2 & 20 \\
\hline
x_1 & 1 & & 1 & & & 4 & 0 \\
x_2 & & 1 & & & & -1 & 2 \\
s_2 & & & -1 & 1 & & -3 & 2 \\
s_3 & & & -1 & & 1 & -4 & 2
\end{array}$$

Optimal! $x_1 = 0$, $x_2 = 2$

value: 20.

<u>Turns out:</u> In above example we solved P, P_1, P_2, P_3, P_4. This can be done by solving P using simplex and 4 dual simplex pivots.

Some more dual Simplex recap.

Assume we want to solve

$$
\begin{array}{c|cc}
 & & \begin{array}{cccccc} & & s_1 & s_2 & & \end{array} & \\
\end{array}
$$

$$
\begin{array}{c}
\\ \\ \\ \\
\end{array}
\left[\begin{array}{c|ccccc|c}
z & & & 7/3 & 2/3 & & 64/3 \\
\hline
x_1 & 1 & & -1/3 & 4/3 & & 8/3 \\
x_2 & & 1 & 1/3 & -1/3 & & 4/3 \\
s_3 & & & -2/3 & -1/3 & 1 & -2/3
\end{array}\right]
\begin{array}{c}
0 \\ 1 \\ 2 \\ 3
\end{array}
$$

Leaving var: s_3

Entering var: pick a non-basic var that has neg. coeff in s_3-row.

Candidates: s_1, s_2

Which one to take?

What happens to z-row if

a) s_1 enters

add $\frac{7/3}{2/3} = \frac{7}{2} \cdot (3)$ to (0)

z-row coeff of s_2: $\frac{2}{3} + \frac{7}{2} \cdot \left(-\frac{1}{3}\right) = \frac{4}{6} - \frac{7}{6} = -\frac{3}{6}$

—— " —— s_3 : $\frac{7}{2}$

b) s_2 enters

add $\frac{2/3}{1/3} = 2 \cdot (3)$ to (0)

z-row coeff of s_1: $\frac{7}{3} - 2 \cdot \frac{2}{3} = 1$

z-row coeff of $s_2 = 2$

If s_2 enters new basis is opt. Not so if s_1 enters.

$\Longrightarrow$ let s_2 enter!

Ex: max $\quad -x_{n+1}$ (IP)

s.t.

$$2x_1 + 2x_2 + \dots + 2x_n + x_{n+1} = n \qquad (*)$$

$$x_i \in \{0,1\} \qquad \forall i \in \{1..n+1\}$$

n: odd no.

- What is the optimal value of (IP)?

Choose $S \subseteq \{1\dots n\}$, $|S| = \frac{n}{2}$ and let

$$x_i = \begin{cases} 1 : & i \in S \text{ or } i = n+1 \\ 0 : & \text{other.} \end{cases}$$

value: -1

- What is the optimal value of LP relaxation?

$x_{n+1} = 0$

pick $S \subseteq \{1\dots n\}$, $|S| = \lfloor \frac{n}{2} \rfloor$

$j \in \{1..n\} \setminus S$

$x_e = 1 \quad \forall e \in S$

$x_j = \frac{1}{2}$

$\Rightarrow$ value 0 !

In the above solution we would branch on x_j.

Create two subproblems by adding either

$$x_j \le 0$$

or $$x_j \ge 1$$

to original constraint set.

Either one of the 2 LP relaxations has optimal value 0.

$\rightarrow$ require further branching!

Consider a subproblem in which k variables have been branched to 1. I.e., $\exists S \subseteq \{1 \dots n\}$, $|S| = k$ s.t.

$\forall i \in S$: $\quad x_i \ge 1 \quad$ is part of current formulation.

$x_i \le 1$ forces: $x_i = 1$ in any feasible soln.

$(*)$ can be rewritten: $\quad 2 \sum_{i \in F} x_i + x_{n+1} = \underbrace{n - 2k}_{\text{odd}}$

F : variables that have not been branched on

LP has value 0 if $\quad |F| \ge \left\lceil \frac{n-2k}{2} \right\rceil$.

$\rightarrow$ set $\left\lfloor \frac{n-2k}{2} \right\rfloor$ vars in F to 1 and 1 var to $\frac{1}{2}$.

<u>Thus:</u> we won't branch on x_{n+1} for $|F| \ge \left\lceil \frac{n-2k}{2} \right\rceil$

If $|F| = \left\lfloor \frac{n-2k}{2} \right\rfloor$ then $x_{n+1} = 1$ and $x_i = 1$ $\forall i \in F$ is only feas. soln. $\rightarrow$ Stop branching.

Need to fix at least $\lfloor \frac{n}{2} \rfloor$ variables in order to stop branching.

$\Longrightarrow$ Tree has depth $\geq \frac{n}{2} - 1$

and $\geq \quad 2 \cdot 2^{\frac{n}{2}-1} - 1 = 2^{\frac{n}{2}} - 1$ nodes.

One more ex:

$$
\begin{array}{lrcl}
\max & 9x_1 + 5x_2 & & \\
\text{s.t.} & 4x_1 + 9x_2 & \leq & 35 \\
& x_1 & \leq & 6 \\
& x_1 - 3x_2 & \geq & 1 \\
& 3x_1 + 2x_2 & \leq & 19 \\
& x & \in & \mathbb{Z}_+^2
\end{array}
$$

Solving LP relax gives: