THE JEEP ONCE MORE OR JEEPER BY THE DOZEN

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1. Introduction. In 1947, N. J. Fine solved the by now famous problem of the jeep [1]. We recall that the problem concerns a jeep which is able to carry enough fuel to travel a distance d, but is required to cross a desert whose distance is greater than d (for example $2d$). It is to do this by carrying fuel from its home base and establishing fuel depots at various points along its route so that it can refuel as it moves further out. It is then required to cross the desert on the minimum possible amount of fuel.

Our purpose here is first to give a very short derivation of the solution of the jeep problem which makes use of a theorem (also famous) of Banach [2]. Second, we consider the situation in which it is required to send a jeep across the desert every day, say, for a week. Of course, having found the best procedure for a single jeep, one could simply repeat this seven times. We shall show, however, that there is a more economical procedure for the case of several jeeps, and in general that the more jeeps one sends across, the lower fuel consumption per jeep. This phenomenon is an example of what economists refer to as "increasing returns to scale," a subject of some economic interest. (It also accounts for the subtitle of this article, for which I apologize herewith.)

2. The problem. To formalize the problem, let us assume that the jeep starts from the origin and moves along the positive x-axis. We choose for the unit of fuel the maximum amount which the jeep can carry, and refer to this unit as a *load*. The unit of distance will then be chosen as the distance the jeep can travel on one load.

Figure 1 below gives a schematic representation of a typical jeep's journey.

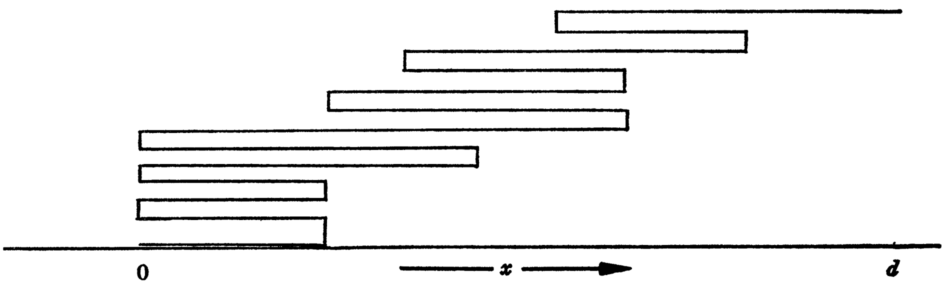

FIG. 1

The wiggly path represents the jeep's travel. Of course, actually the path lies entirely on the x-axis. It has been stretched vertically simply to make it visible. Note that because of our choice of units, the length of this path is precisely equal to the amount of fuel consumed. In the figure, the jeep reaches a point at dis-

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tance d from the origin. The original jeep problem asks for the minimum fuel consumption (hence path length), which will allow the jeep to reach this point.

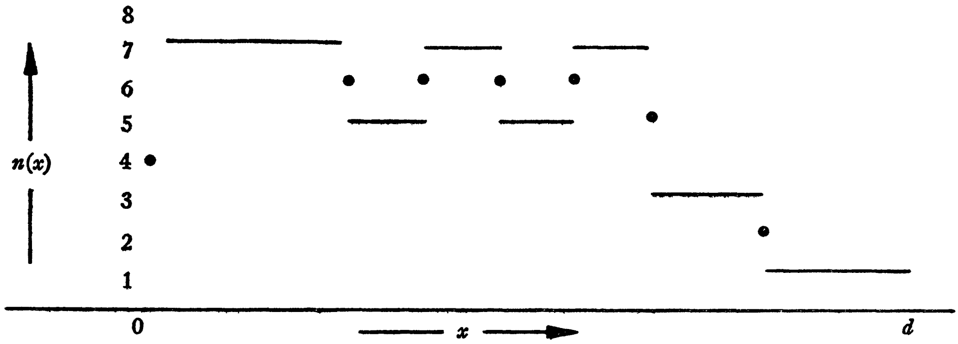

FIG. 2

It turns out to be somewhat more convenient to turn the problem around (see e.g. [3]) and obtain a formula for the function $d(f)$ giving the farthest point which the jeep can reach on f loads of fuel. Our first aim is to prove the formula

$$d(f) = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2f-1}\cdot \tag{1}$$

The key idea in our solution is to make use of a formula of Banach for the path length of a curve in one dimensional space. (Path length in one-space is usually referred to as *total variation*; we prefer the geometric terminology as being more suggestive in the present context.) To utilize Banach's Formula we define for each point x on the interval $[0, d]$, the *valance* $n(x)$ as the number of times during its journey that the jeep is at the point x. Figure 2 above gives the graph of the valance $n(x)$ corresponding to the jeep's journey plotted in Figure 1. Strictly speaking, if one allowed very general paths, $n(x)$ might be infinite for some points. This would not affect Banach's Formula, which states

$$\text{total path length} = \int_0^d n(x)dx.$$

Of course, Banach was not concerned with jeeps. He considered a continuous real valued mapping $x=\phi(t)$, and $n(x)$ was simply the number of points which mapped onto the point x, or the cardinality of $\phi^{-1}(x)$. We can also think of the problem this way if we take the wiggly line of Figure 1 to be the graph of the jeep's position plotted, say, as a function of the time.

For "reasonable" paths, Banach's Formula is obvious. A reasonable path for a jeep is one with a finite number of points at which it reverses direction (it would be a remarkable jeep indeed that could execute an unreasonable path). To prove the formula, partition $[0, d]$ into sets $X_1, X_2, \cdots$, where

$$X_k = \{x \mid n(x) = k\}.$$

Because of reasonableness, there are only finitely many nonempty X_k, and each of them is a union of disjoint intervals. (Some of them may consist of single points. In the case of the jeep's tour, this will be the case whenever k is even. Do you see why?) Over each interval of X_k lie exactly k intervals of the jeep's path. Therefore,

$$\text{total path length} = \sum_{k=1}^{\infty} k \text{ (length of } X_k).$$

But the term on the right is precisely the definition of $\int_0^d n(x)dx$. (Notice that this is the definition of the integral in the sense of Lebesgue rather than Riemann! Of course for continuous functions, and in particular for reasonable functions, the two concepts are equivalent.) We remark that Banach's formula holds for unreasonable as well as reasonable paths (though the general proof is fairly involved), and hence our solution of the jeep problem will hold for unreasonable as well as reasonable jeeps.

3. The solution for one jeep. We return now to the problem of computing $d(f)$, and assume for the moment that f is an integer. We wish to determine how far the jeep can get on f loads of fuel. For any jeep's tour, we define the sequence of points $x_0, x_1, \cdots, x_f$ on the interval $[0, d]$ where $x_f=0$, $x_0=d$, and in general x_k is the point such that the total path length (hence fuel consumption) to the right of x_k is exactly k units. Clearly the points x_k form a strictly decreasing sequence and there will be exactly one unit of path length between x_{k+1} and x_k. The basic observation we need is the following:

LEMMA 1. *If* $x<x_k$, *then*

$$n(x) \geqq 2k+1. \tag{2}$$

Proof. Since x is to the left of x_k, the jeep must consume more than k loads of fuel to the right of x. Since the jeep can only carry one load at a time, it must therefore cross the point x at least $k+1$ times from the left. But between any two crossings from the left there must be a crossing from the right, so there must be at least k crossings from the right. Then the jeep must arrive at the point x at least $2k+1$ times, which is what the lemma asserts.

We now combine this result with Banach's Formula and get

$$\begin{aligned} 1 &= (\text{path length between } x_{k+1} \text{ and } x_k) \\ &= \int_{x_{k+1}}^{x_k} n(x)dx \geqq (2k+1)(x_k - x_{k+1}) \end{aligned}$$

so $x_k - x_{k+1} \leqq 1/(2k+1)$. Summing from 0 to $f-1$ implies

$$\sum_{k=0}^{f-1} (x_k - x_{k+1}) = x_0 - x_f = d \leqq 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2f-1}, \tag{3}$$

which gives an upper bound for $d(f)$. It remains only to show that this bound can be achieved, and this is easily done by induction. The formula is clearly correct when $f=1$. Suppose now we are allowed $f+1$ loads. Then let the jeep take $f+1$ loads to the point $1/(2f+1)$. This will involve $f+1$ outward trips and f return trips, hence $2f+1$ trips of length $1/(2f+1)$; therefore a total of one load will be consumed leaving f loads deposited at the point $1/(2f+1)$. By the induction hypothesis, the formula of (1) holds from this point on, completing the proof.

For the case where f is not an integer, the same type of argument shows that

$$d(f) = 1 + \frac{1}{3} + \cdots + \frac{1}{2[f]-1} + \frac{\{f\}}{2[f]+1}, \tag{4}$$

where $[f]$ and $\{f\}$ are the integral and fractional part of f respectively. In other words, one simply interpolates linearly between integral values of f. The graph of $d(f)$ is plotted in Figure 3.

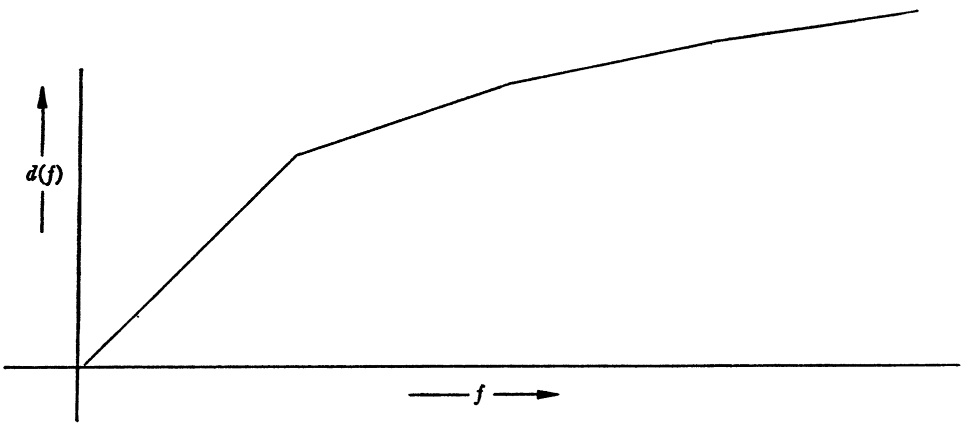

FIG. 3

Since the odd harmonic series diverges, it follows that a desert of any size can be crossed.

I should remark that after finding the solution just given, I became aware of the rather similar one given in [4]. The use here of the Banach Formula seems to tighten the argument somewhat and make the reasoning more transparent. Also we do not need to assume *a priori* that there will be only a finite number of depots. It is at least conceivable that the optimal solution would involve infinitely many deposits, or even a sort of continuous smear of fuel spread out along the route. One could no doubt formulate a very general problem in terms of measure theory. The argument above shows, however—thanks again to Banach's Formula—that this more general behavior could not give any improvement in fuel consumption.

Before leaving the single jeep, we consider the case in which the jeep is required to cross the desert and then return. For this case the arguments are the same as before except that (2) becomes

$$n(x) \geqq 2k + 2. \tag{5}$$

Letting $\tilde{d}(f)$ be the longest possible round trip (e.g., twice the distance to the farthest point), we get the even simpler formula

$$\tilde{d}(f) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{f}, \tag{6}$$

where as before equality can be achieved. We note the familiar fact that a round trip can be substantially cheaper than two one way trips. In fact comparing (6) with (1), we have

$$d(f) - \frac{1}{2}\tilde{d}(f) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{2f};$$

but this is bounded by $\sum_{n=1}^{\infty} (-1)^{n+1}(1/n) = \log 2$. Thus for long distance, the increase in fuel cost for a round trip as against a one way trip becomes negligible.

4. Several jeeps. We turn now to the problem of m jeeps, and we shall compute the function $d_m(f)$ giving the most distant point which all jeeps can reach if they have f loads of fuel to share between them. In other words, the sum of the path lengths of all of the jeeps must not exceed f.

We proceed by defining the points x_k exactly as in the one jeep case, as the point to the right of which the total combined path length of all jeeps is k. The analogue of Lemma 1, however, is slightly more complicated.

LEMMA 2. *For any x in $[0, d]$, $n(x) \geqq m$. If $x < x_{m+r}$ ($r \geqq 0$) then*

$$n(x) \geqq m + 2r + 2. \tag{7}$$

Proof. The first assertion simply corresponds to the fact that all m jeeps are required to reach the point d. Concerning inequality (7), we see that more than $m+r$ loads must be transported beyond the point x, so this point must be crossed at least $m+r+1$ times from the left. But since there are only m jeeps, in order to achieve $m+r+1$ crossings from the left there must be at least $r+1$ crossings from the right, giving $m+2r+2$ crossings in all.

We proceed to calculate $d_m(f)$. Notice that for $f \leqq m$, the problem is trivial and $d_m(f) = m/f$ (why?) so we assume that $f = m+s$, where $s > 0$. Again restricting ourselves to integral values of s, we claim

$$\begin{aligned} d_m(f) &= 1 + \frac{1}{m+2} + \frac{1}{m+4} + \cdots + \frac{1}{m+2s} \\ &= 1 + \frac{1}{m+2} + \frac{1}{m+4} + \cdots + \frac{1}{2f-m}, \end{aligned} \tag{8}$$

for the path length from x_{k+1} to x_k is one, so

$$1 = \int_{x_{k+1}}^{x_k} n(x)dx \geqq m(x_k - x_{k+1}), \qquad \text{for } k < m,$$

and

$$1 = \int_{x_{m+r+1}}^{x_{m+r}} n(x)dx \geqq (m + 2r + 2)(x_{m+r} - x_{m+r+1}).$$

Therefore $x_k - x_{k+1} \leqq 1/m$ for $k<m$, and

$$x_{m+r} - x_{m+r+1} \leqq \frac{1}{m + 2r + 2} \qquad \text{for } r \geqq 0.$$

Summing for $0 \leqq k \leqq m-1$ and $0 \leqq r \leqq s-1$ gives (8) as an inequality. Again an inductive proof shows that equality can be achieved. Assuming the formula correct for $m+s$ loads of fuel, we see that $m+s+1$ loads can be moved to the point $1/(m+2s+2)$ by having one jeep make $s+1$ round trips and then having all m of them make the one way trip to this point. This will use up one load, so that $m+s$ loads are deposited, and the induction can be continued. Note that there are various ways in which this optimal journey can be performed. One way is to have one of the jeeps do all the work of setting up the various depots, while the others simply move outward without turning around, refueling as they go. It is, however, not possible to prescribe the same path for all jeeps, for if all jeeps followed the same route, then the function $n(x)$ would have to be divisible by m at all points, which it clearly is not in an optimal trip. We shall return to this point later.

To prove the result about increasing returns, we wish to compare the distance which one jeep goes on f loads with that which m jeeps go on mf loads. For this purpose we rewrite (8) for the case $s = m(f-1)$ as follows:

$$\begin{aligned} d_m(mf) = 1 &+ \left(\frac{1}{m+2} + \cdots + \frac{1}{3m}\right) + \left(\frac{1}{3m+2} + \cdots + \frac{1}{5m}\right) + \cdots \\ &+ \left(\frac{1}{2(f-3)m+2} + \cdots + \frac{1}{(2f-1)m}\right), \end{aligned} \tag{9}$$

where each term in parentheses contains m summands and there are f terms in all. We may also rewrite (3) as

$$\begin{aligned} d(f) = 1 &+ \left(\frac{1}{3m} + \frac{1}{3m} + \cdots + \frac{1}{3m}\right) + \left(\frac{1}{5m} + \cdots + \frac{1}{5m}\right) + \cdots \\ &+ \left(\frac{1}{(2f-1)m} + \cdots + \frac{1}{(2f-1)m}\right), \end{aligned} \tag{10}$$

where there are also m summands in each term in parentheses. A term by term comparison between (9) and (10) shows the advantage of using many jeeps.

For the special case $m=2$, we get

$$d_2(2f) - d(f) = \left(\frac{1}{4} - \frac{1}{6}\right) + \left(\frac{1}{8} - \frac{1}{10}\right) + \cdots + \left(\frac{1}{4f-4} - \frac{1}{4f-2}\right)$$
$$= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots + \frac{1}{2f-2} - \frac{1}{2f-1}\right).$$

Thus the longer the trip the greater the saving. On the other hand since the terms in parentheses above are part of a convergent series, it follows that the total amount which is saved remains bounded as the trip gets longer.

For the round trip problem, the formula is again simpler. Let $\tilde{d}_m(f)$ be the round trip distance which m jeeps can achieve on f loads. Then

$$\tilde{d}_m(mf) = 1 + \frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{mf}\cdot \tag{11}$$

So again, the more jeeps there are, the further out they can get on the same amount of fuel; or returning to the original problem, the less fuel per jeep is needed to reach a preassigned point. But there is a limit to the advantage one obtains in increasing the number of jeeps; for as m approaches infinity, a standard calculation (comparison with $\int dx/x$) shows

$$\lim_{m\to\infty} \tilde{d}_m(mf) = 1 + \log f, \tag{12}$$

so no matter how many jeeps there are, one cannot go further than $1+\log f$ on f loads of fuel per jeep. In terms of the original problem, we get in the limit

$$f = e^{d-1} \tag{13}$$

so the amount of fuel needed increases exponentially with the length of the desert, as one might have guessed.

The multi-jeep round trip problem can be interpreted in another way. Instead of thinking of m jeeps, each making a round trip, one may consider a single jeep which must make round trips, say on m successive days. The solution here is the same as for the m jeep problem with the consequent saving of fuel. In fact, on the first day the jeep can set up all the depots it will need for the following days, as one easily sees.

5. Some Final Remarks and Questions. The last paragraph above raises an interesting question. Suppose instead of being concerned about m round trips across the desert, one has decided to go into the desert crossing business and plans to make a round trip daily into the indefinite future. What sort of routine should one then use? As we have just seen, it would be uneconomical to use the single round trip routine each day. Similarly, repeated use of the m-day routine would be inferior to using the $(m+r)$-day routine, so that no periodic program of this sort could be optimal. On the other hand, if one is to consider programs which are not periodic, it is no longer clear what one should mean by an optimal program. In any case, it appears that the problem of finding the best "steady state" routine has no exact solution at all, so that in practice one would have to settle for a routine that was "almost optimal."

There are many other jeep problems one can think of. Helmer [5] has considered some fairly complicated situations in which the number of depots one is allowed to establish is limited. To get a feeling for this sort of problem, the reader might look at the problem of crossing a desert of length 2 when only 3 intermediate depots are permitted.

An apparently simple question is the round trip problem in which fuel is available at both ends of the desert, but I must confess with embarrassment that I have not been able to find the solution. It is not hard to see that one can do at least as well in this case as in the case of two jeeps making one-way trips, but it may be possible to do better. The difficulty here as with many optimization problems is that there does not appear to be any simple way to determine whether or not a given solution is optimal. The upper bound given by Banach's Formula does not seem to be available for this case. I put this problem forward as a challenge to jeepologists.

I conclude with some historical remarks. Shortly after the publication of Fine's solution, Phipps [3] derived the same result by arguing that the single jeep problem is equivalent to a problem involving a convoy of jeeps which travel together, some being used to refuel others. The solution to the convoy problem is very simple, but the argument that this problem is equivalent to the original problem does not seem to be quite complete. Using this equivalence, however, one can also easily derive the result given here on increasing returns. Finally, there seems to be a feeling among many people that the jeep problem can be solved by the functional equation method of dynamic programming. In fact the problem occurs as an exercise in the book of Bellman [6], but the solution is not given there and I know of no way of solving the problem by this method.

6. Addendum. It has been pointed out to me that if one accepts the convoy equivalence of Phipps, then dynamic programming can be used (see e.g. [7]). However, for the convoy problem, the solution is almost obvious anyway. Imagine that f jeeps set out. Since all but the last must return home, we suppose that $f-1$ of them consume fuel at the rate 2 and the last one, the "red, white, and blue" jeep that will make the final crossing, consumes at the rate 1. Thus, initially, the convoy is consuming at the rate $2f-1$, which it does until one load has been consumed, after which the first jeep can be abandoned. The remainder then go on consuming at the rate $2f-3$, etc.

I should also remark that for Helmer's variation in which the number of depots is specified, dynamic programming does seem to be the appropriate tool to use.

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MATHEMATICAL NOTES

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A NOTE ON SECOND CATEGORY TOPOLOGICAL GROUPS

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In this note we prove that if G is a topological group and H is the closure of the identity, then G is of second category if and only if G/H is of second category. Consequently since G/H is a Hausdorff group the study of second category topological groups can in most cases be restricted to Hausdorff topological groups.

Let X be a topological space. A subset A of X is *nowhere dense* in X if the interior of $\bar{A}$ (the closure of A) is empty. A subset A of X is said to be of *first category* in X if it is the union of a countable family of nowhere dense sets. If a subset A of X is not of first category, then it is said to be of *second category* in X. Throughout this paper the topology of G/H will be the quotient topology.

We begin by giving some well-known facts concerning topological groups.

LEMMA 1. *Let G be a topological group and H a normal subgroup of G and $\phi: G \to G/H$ the natural homomorphism, then:*

a. *The mapping $\phi: G \to G/H$ is a continuous and open mapping.*

b. *If H is compact, then $\phi: G \to G/H$ is a closed mapping.*

c. *If e is the identity of G and $H = \overline{\{e\}}$, then H is a compact closed normal subgroup of G and G/H is Hausdorff.*

Proof. The following page references are all in [1]. For proofs of (a) and (b) see pages 36 and 37. For (c), that H is a closed normal subgroup is on page 33 and that G/H is Hausdorff is on page 45. By Proposition 3, Section 20 of [3]

$$H = \overline{\{e\}} = \cap U_\alpha, \quad \text{where} \quad \{U_\alpha\}$$