C&O 367/CM 442 Nonlinear Optimization – Winter 2009

Assignment 2 Due date: Wednesday Jan. 28, 2009

Assignments are due before the start of class on the due date.
Write your name and ID# clearly, and underline your last name.

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C&O 367
Assignment 1

Due on Thursday, Jan. 14 (before start of class)
Instructor H. Wolkowicz

1 Quadratic Forms and Projections —— 9 Marks

An $n \times n$ matrix P is called a *projection* matrix if $P^T = P$ and $PP = P$. Prove that if P is a projection matrix, then

1. $I - P$ is a projection matrix.
2. P is positive semidefinite.
3. $||Px|| \leq ||x||$, for any x, where $|| \cdot ||$ is the Euclidean norm.

2 Differential Geometry

Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable. Let $\bar{x} \in \mathbb{R}^n$ with $\nabla f(\bar{x}) \neq 0$.

2.1 Contours and Tangent Planes —— 6 Marks

Let $C = \{x : f(x) = f(\bar{x})\}$ be the contour (level curve) of f at $\bar{x}$, and let T denote the tangent plane to the level curve at $\bar{x}$. Show that the two direction vectors

$$\pm \frac{1}{||\nabla f(\bar{x})||} \nabla f(\bar{x})$$

are orthogonal to the contour C at $\bar{x}$, i.e. show that they are orthogonal to the tangent plane T at $\bar{x}$.

2.2 Direction of Steepest Ascent —— 4 Marks

Show that the direction vector $s = \frac{1}{||\nabla f(\bar{x})||} \nabla f(\bar{x})$ has the greatest slope, over all vectors for which $s^T s = 1$. (The slope refers to the directional derivative.)

3 Convex Sets

3.1 Halfspaces —— 5 Marks

When does one halfspace contain another? More precisely, give conditions on $a, \bar{a}, b, \bar{b}$ under which

$$\{x : a^T x \leq b\} \subseteq \{x : \bar{a}^T x \leq \bar{b}\},$$

where both $a \neq 0, \bar{a} \neq 0$. Also, find conditions under which the two halfspaces are equal.

3.2 Solution Set of a Quadratic Inequality —— 5 Marks

Let $F \subseteq \mathbb{R}^n$ be the solution set of a quadratic inequality,

$$F = \{x \in \mathbb{R}^n : x^T A x + b^T x + c \leq 0\},$$

where $A \in \mathcal{S}^n$, the set of $n \times n$ symmetric matrices, $b \in \mathbb{R}^n$, and $c \in \mathbb{R}$.

Show that F is convex if and only if $A \succeq 0$ (is positive semidefinite).

4 Convex Functions

4.1 Pointwise Maximum and Supremum

Show that the following functions $f : \mathbb{R}^n \to \mathbb{R}$ are convex.

4.1.1 Maximum of Norms —— 5 Marks

$f(x) = \max_{i=1,\ldots,k} \|A^{(i)}x - b^{(i)}\|_2$, where $A^{(i)} \in \mathbb{R}^{m \times n}, b^{(i)} \in \mathbb{R}^m$.

4.1.2 Largest Components —— BONUS: 5 Marks

$f(x) = \sum_{i=1}^{r} |x|_{[i]}$ on $\mathbb{R}^n$, where $|x|$ denotes the vector with $|x|_i = |x_i|$ (i.e. —x— is the absolve value of x, componentwise), and $|x|_{[i]}$ is the ith largest component of $|x|$. In other words, $|x|_{[1]}, |x|_{[2]}, \ldots, |x|_{[n]}$ are the absolute values of the components of x, sorted in nonincreasing order.