C&O 367: Optimality Conditions and Duality, Assignment 5

Due on Friday, April 4, 2008.

Contents

1 Preliminaries

This assignment extends the results presented in class for the convex program, (CP), to include the case where (CP) can have affine equality constraints. We first recall the following definitions for this new case.

1. Convex Program:

$$(CP) \qquad \begin{array}{rll} \min & f(x) & \\ \text{s.t.} & g_k(x) \le 0, & k = 1, \ldots, m, \\ & h_j(x) = 0, & j = 1, \ldots, p, \end{array}$$

   where the functions f, g_k, h_j are all real valued (sufficiently smooth) convex functions defined on $\Omega \subset \mathbb{R}^n$, the functions $-h_j$ are also convex for all j, (i.e. the functions $\pm h_j$ are both convex and concave, $\forall j$) and Ω is an open convex set.

   (a) The *feasible set* of (CP) is

   $$\mathcal{F} := \{x \in \Omega : g_k(x) \le 0, \forall k, \text{ and } h_j(x) = 0, \forall j\}.$$

   The *active set* of (CP) at $\bar{x}$ is $\mathcal{A}(\bar{x}) := \{k : g_k(\bar{x}) = 0\}$.
   The *generalized Slater constraint qualification* for (CP), denoted (GCQ), is:
   $\exists \hat{x} \in \Omega$ such that $g_k(\hat{x}) < 0, \forall k$ and $h_j(\hat{x}) = 0, \forall j$.

   (b) For $\bar{x} \in \mathcal{F}$, the *tangent cone of* $\mathcal{F}$ *at* $\bar{x}$ is

   $$T_{\mathcal{F}}(\bar{x}) := \overline{cone(\mathcal{F} - \bar{x})},$$

   i.e. it is the closure of the convex cone generated by the set $\mathcal{F} - \bar{x}$.

   (c) For $\bar{x} \in \mathcal{F}$, the *linearizing cone of (CP) at* $\bar{x}$ is

   $$L(\bar{x}) := \left\{d \in \mathbb{R}^n : d^T \nabla g_k(\bar{x}) \le 0, \forall k \in \mathcal{A}(\bar{x}), d^T \nabla h_j(\bar{x}) = 0, \forall j\right\}.$$

2 Problems on Definitions of (CP)

1. Show that the feasible set $\mathcal{F}$ is a convex set. Is it a closed set as well? If not, are there cases when it is a closed set?

2. Let $\bar{x} \in \mathcal{F}$ and

   $$\mathcal{D}(\bar{x}) := \{d \in \mathbb{R}^n : g_k(\bar{x} + \bar{\alpha} d) \le 0, \forall k \in \mathcal{A}(\bar{x}), \text{ and } h_j(\bar{x} + \bar{\alpha} d) = 0, \forall j, \text{ for some } \bar{\alpha} > 0\}.$$

   Show that $T_{\mathcal{F}}(\bar{x}) = \{0\} \cup \overline{\mathcal{D}(\bar{x})}$.

3. Let $\bar{x} \in \mathcal{F}$. Show that the linearizing cone $L(\bar{x})$ is the polar cone of a finite set, i.e.

   $$L(\bar{x}) = S^+ := \{v_1, \ldots, v_t\}^+; \qquad (1)$$

   and state what the vectors v_i in S are.

3 Rockafellar-Pshenichnyi Optimality Conditions for (CP)

State and prove the RP characterization of optimality for (CP), i.e. the optimality conditions for $\bar{x} \in \operatorname{argmin}_{x\in\mathcal{F}} f(x)$.

4 Generalized Farkas' Lemma

Consider the linear system

$$\begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = b, \quad \lambda \in \mathbb{R}^{t_m}_+ \text{ nonnegative and, } \mu \in \mathbb{R}^p \text{ free,} \tag{2}$$

where the matrices A and B are $n \times t_m$ and $n \times p$, respectively. And consider the implication

$$\left\{ \begin{pmatrix} A & B \end{pmatrix}^T d = \begin{pmatrix} p \\ z \end{pmatrix}, p \in \mathbb{R}^{t_m}_+, z = 0 \in \mathbb{R}^p \right\} \implies b^T d \geq 0. \tag{3}$$

State and prove a generalized version of Farkas' Lemma based on (2) and (3), i.e. (2) *holds if and only if* (3) *holds.*
(**Hint:** You can use a result proved in a previous assignment.)

5 Karush-Kuhn-Tucker Optimality Conditions for (CP)

State the KKT optimality conditions for (CP) at a point $\bar{x}$. State carefully when necessity holds and when sufficiency holds. In particular, state when a constraint qualification (CQ) is needed and state an appropriate (CQ).

6 Duality for Quadratic Programs

Let the matrices: Q be $n \times n$ symmetric, positive semidefinite; A be $m \times n$; and B be $p \times n$. let g, a, b be vectors of appropriate dimension. Derive a (Wolfe type) dual program for the (convex) quadratic program

$$\min\{q(x) := \frac{1}{2} x^T Q x + g^T x : Ax \leq a, Bx = b\}.$$

Then, show that strong duality holds.

7 BONUS Questions

7.1 BONUS: Proof and (CQ) for KKT Optimality Conditions

1. Prove the above KKT optimality conditions at $\bar{x}$. In particular, state and use an appropriate weakest constraint qualification, WCQ.
   (**Hint:** Use Items 1b) and1c) in Section 1 for the WCQ. then use (1) and the generalized Farkas Lemma to connect the KKT and the RP optimality conditions.)
2. Show that the generalized Slater constraint qualification, (GCQ), is a valid (CQ) for (CP), by using the above WCQ.

7.2 BONUS: Applications for Eigenvalue Bounds

Let S be a given $n \times n$ symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Let $m := \operatorname{trace} \mathrm{S}$ and $s^2 := \frac{\operatorname{trace} \mathrm{S}^2}{n} - m^2$. Recall the convex program solved in class

$$\begin{array}{rccl} v_p := \min & \lambda_n & & \\ \text{s.t.} & \sum_{i=1}^n \lambda_i & = & \operatorname{trace} \mathrm{S} \\ & \sum_{i=1}^n \lambda_i^2 & \leq & \operatorname{trace} \mathrm{S}^2. \end{array} \tag{4}$$

The optimal solution of this program resulted in the lower bound on the smallest eigenvalue of S, i.e.

$$\lambda_n \geq m - \sqrt{n-1}s.$$

By adding the appropriate constraints $\lambda_i \geq \lambda_n, i = 1, \ldots, n-1$ and changing the objective function appropriately, in (4), derive and prove a lower bound on the second smallest eigenvalue λ_{n-1}.