Assignment 4, C&O 367, W08

This assignment has 4 problems and 2 pages.

Due Fri. Mar. 14, 2008

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1 Lagrange Multipliers

1. Find the extrema of $f(x) = x_1 x_2^2 x_3^3$ subject to $x_1^2 + x_2^2 + x_3^2 = 1$.

2. Find the extrema of the function of two variables $x_3 = f(x_1, x_2)$ that is defined implicitly by the relation

$$F(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - x_1 x_3 - x_2 x_3 + 2x_1 + 2x_2 + 2x_3 - 2 = 0.$$

3. Show that the Lagrange multiplier rule suggests the wrong solution $x = (9, 4)$ for the problem

$$\max 2x_1 + 3x_2 \text{ subject to } \sqrt{x_1} + \sqrt{x_2} = 5.$$

   Explain why.

2 Model Convex Program

1. Model the problem to find the point closest to $(2, 3)$ with sum of the coordinates not larger than 2 and with first coordinate not larger than 2 in absolute value.

2. Draw the feasible set of the problem in Item 1 and obtain the optimal solution geometrically.

3 Problems from Text

1. Page 212, #2,#3; Page 213 #6.

2. Page 214, #13. Also prove the converse, i.e. $(C^*)^* = C$ implies that C is a closed (convex) cone.

4 Hölder's Inequality (BONUS Question worth 5 marks for midterm.)

For real $p > 1$, define q by $p^{-1} + q^{-1} = 1$, and for $x \in \mathbb{R}^n$ define (the p-norm)

$$\|x\|_p = \left(\sum_1^n |x_i|^p \right)^{1/p}.$$

For a nonzero vector y in $\mathbb{R}^n$, consider the optimization problem

$$\inf \left\{ \langle y, x \rangle \mid \|x\|_p^p \le 1 \right\}, \tag{1}$$

where $\langle y, x \rangle$ denotes the standard inner-product (dot product) in $\mathbb{R}^n$.

1. Prove $\frac{d}{du}|u|^p/p = u|u|^{p-2}$, for all real u.
2. Prove that the reals u and v satisfy $v = u|u|^{p-2}$ if and only if $u = v|v|^{q-2}$.
3. Prove that problem (1) has a nonzero optimal solution.
4. Use the Karush-Kuhn-Tucker conditions to find the unique optimal solution. (Note that the KKT conditions for (1) are the same as the optimality conditions using Lagrange multipliers except that the Lagrange multiplier is restricted to be nonnegative, $\lambda \ge 0$.)
5. Deduce that any vectors x and y in $\mathbb{R}^n$ satisfy $\langle y, x \rangle \le \|y\|_q \|x\|_p$.