Assignment 3, C&O 367, W08

This assignment has 7 problems and 3 pages.

Due Wed.Feb. 27, 2008

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1 MATLAB (Roundoff Error)

1. Pretend you have a computer with base 10 and precision 4 that truncates after each arithmetic operation; for example, the sum of $24.57 + 128.3 = 152.87$ becomes 152.8. What are the results when $128.3, 24.47, 3.163$, and 0.4825 are added in ascending order and descending order in this machine? How do these compare with the correct ("infinite-precision") result? What does this show you about adding sequences of numbers on the computer?

2. Now consider the quadratic equation
$$x^2 - 100x + 1 = 0.$$
Use the formula for the solution of a quadratic equation to find the roots of this equation. (Assume again that you have the same computer as above, but with precision 5.) How many correct digits do you get? Why? Can you improve the accuracy by changing the algorithm? (Recall that the product of the two roots of a quadratic equals the ratio of the last and first coefficient, i.e. $x_1 x_2 = c/a$, where here a, c both equal 1.)

2 MATLAB (Newton's Method for Root Finding)

1. Write a MATLAB program to implement Newton's method for root finding. Apply it to the polynomials
$$p(x) = x^3 - 3x - 2; \qquad q(x) = x^2 - x - 2.$$
starting at the points: $x = 5.95, x = 2.95, 0, -2$.

   Compare your results with those you obtain using the MATLAB optimization toolkit, and with the MATLAB commands roots and fzero. Which roots do you converge to? Why?

3 Newton's Method Convergence

1. Classify the (rates of) convergence of the following sequences:

   (a)
   $$x_n = e^{(-n^2)}$$

   (b)
   $$1\ ,\ 0\ ,\ 1/2\ ,\ 0\ ,\ 1/4\ ,\ 0\ ,\ 1/8$$

   (c)
   $$x_n = \frac{1}{n}a^n, \quad (0 < a < 1)$$

   (d)
   $$x_n = a^{\log n} \quad (0 < a < 1)$$

   (e)
   $$x_n = a^{n \log n} \quad (0 < a < 1)$$

2. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable. Prove that Newton's method converges linearly to a multiple root, i.e. a root where $f(x^*) = f'(x^*) = 0$. (Assume that you obtain an infinite sequence of iterations for Newton's method.)

4 Hessians and Convex Functions

1. Consider the function $f(x) = \frac{1}{x_1} + e^{x_2} - \sum_{i=1}^{3} \log(x_i)$ defined for $x_i > 0, \forall i$. Use the Hessian to show that this function is convex.

2. Consider the matrix $A = \begin{pmatrix} 21 & -9 & 6 & 4 \\ -9 & 6 & -2 & -2 \\ 6 & -2 & 4 & 0 \\ 4 & -2 & 0 & 4 \end{pmatrix}$. Using the Cholesky factorization, verify whether or not A is positive definite. If A is positive definite, confirm the result using the leading principle minors.

5 Eigenvalue and Optimization

(Text: Based on Problem 31, page 36)

1. Let A be an $n \times n$ symmetric matrix, and let $\lambda_1 \geq \ldots \geq \lambda_n$ denote its n eigenvalues. Diagonalize A to show that (based on the Raleigh quotient)

$$\min_i \lambda_i = \min_{\|x\|=1} x^t A x; \quad \max_i \lambda_i = \max_{\|x\|=1} x^t A x.$$

2. Show that the quadratic form $Q_A(x) = x^t A x$ is coercive if and only if A is positive definite.

6 Optimality

Find the gradient and Hessian of

$$f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2.$$

Verify that $x^* = (1, 1)^T$ is the minimizer. Show that the Hessian $H(x)$ is singular if and only if x satisfies the condition

$$x_2 - x_1^2 = 0.005.$$

Hence show that $H(x)$ is positive definite for all x such that $f(x) < 0.0025$.

7 Steepest Descent

Consider the following two-variable problem

$$\max f(x) = 2x_1x_2 + 2x_2 - x_1^2 - 2x_2^2.$$

Perform several (3-4) iterations of steepest ascent using an exact line search. Start with the initial point (0,0). What is the optimal solution? (You can use MATLAB to help with the iterations but carefully show the steps of steepest descent.)