C&O 367 Assignment 1

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Due on Wed. Jan. 16, 2008

Note: Please explain your answers and proofs carefully. A yes or no does not constitute a valid answer; nor does a numerical value with no explanation constitute a valid answer.

1 GEOMETRY

Definition 1 *For a vector* $x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n$*, we define the* (Euclidean) norm *and* (Euclidean) inner product

$$\|x\| := (x_1^2 + \cdots + x_n^2)^{1/2},$$

$$\langle x, y \rangle := x \cdot y := x_1 y_1 + \cdots + x_n y_n.$$

Theorem 1 *(Cauchy-Schwarz)* $|x \cdot y| \leq \|x\| \|y\|$*, with equality if and only if* $x = \lambda y$*, for some* λ.

1.1 EXERCISES

1. (10) Explain the geometrical significance for the vectors x and y of:

   (a)
   $$x \cdot y = 0.$$

   (b)
   $$x \cdot y > 0.$$

   (c)
   $$\frac{x \cdot y}{\|x\| \|y\|} \geq \frac{1}{\sqrt{2}}.$$

2. (10) Prove that the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(x) = a \cdot x + \alpha$ is continuous, where $a \in \mathbb{R}^n$ is a given vector and $\alpha \in \mathbb{R}$ is a given scalar.

2 CALCULUS

For a function $g : \mathbb{R}^n \to \mathbb{R}$, the *gradient* is denoted

$$\nabla g(x) := \begin{pmatrix} \frac{\partial g(x)}{\partial x_1} \\ \frac{\partial g(x)}{\partial x_2} \\ \cdots \\ \frac{\partial g(x)}{\partial x_n} \end{pmatrix}$$

2.1 EXERCISES

1. (5) If $g(x) = ||x||$, calculate $\nabla g(x)$.

2. (10) Suppose $g : \mathbb{R}^n \to \mathbb{R},\ a, b \in \mathbb{R}^n$, and $f(t) := g(a + tb)$. Calculate $f'(t)$.

2.2 EXERCISES

Definition 2 *Suppose* $f : \mathbb{R} \to \mathbb{R}$ *is infinitely differentiable at* $x = a$. *The* Taylor Series *of* f *about* a *is:*

$$f(a) + f'(a)(x - a) + \frac{1}{2!} f''(a)(x - a)^2 + \frac{1}{3!} f'''(a)(x - a)^3 + \cdots$$

Write down the Taylor series of:

1. (5)
$$f(x) = x^3,\ \text{about } x = 1.$$

2. (5)
$$f(x) = \log(1 + x),\ \text{about } x = 0.$$

3 TOPOLOGY

Definition 3 *The* open ball $B(x; r) := \{y \in \mathbb{R}^n : ||x - y|| < r\}$. *Suppose that* D *is a subset of* $\mathbb{R}^n$.

Interior: $x \in \operatorname{int} D$ *if there exists* $r > 0$ *with* $B(x; r) \subset D$.

Closure: $x \in \operatorname{cl} D$ *if there exists a sequence* $x^k \in D$ *with* $x^k \to x$.

Boundary: $x \in \partial D$ *if* $x \in \operatorname{cl} D \backslash \operatorname{int} D$.

D *is* open *if* $D = \operatorname{int} D$. D *is* closed *if* $D = \operatorname{cl} D$.

3.1 EXERCISES

1. (15) For each of the following sets, find the interior, the closure, and the boundary. Then determine which of the sets are open, closed, neither, or both.

   (a)
   $$\{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0\}.$$

   (b)
   $$\{(x_1, x_2) : x_1 > 0, x_2 > 0\}.$$

   (c)
   $$\{(x_1, x_2) : x_1 > 0, x_2 \geq 0\}.$$

   (d)
   $$\mathbb{R}^n$$

   (e)
   $$\left\{(x_1, x_2) : x_1^2 + x_2^2 < 0\right\}.$$

   (f)
   $$\emptyset.$$

2. (a) (10) Prove that D is closed if and only if the complement D^c is open.

   (b) (10) Prove that $x \in \partial D$ if and only if for any $r > 0$ there exists a $y \in B(x; r) \cap D$ and a $z \in B(x; r) \cap D^c$.

4 MATRICES

4.1 EXERCISES

1. (10) Let

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

   (a) Calculate the determinant of A.

   (b) Calculate the rank of A.

   (c) What is the rank of A^T.

2. (10) Let

$$B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

   Calculate the eigenvalues and eigenvectors of B.

3. (10) Let

$$C = \begin{pmatrix} 3 & 1 & 1 & 4 \\ \rho & 4 & 10 & 1 \\ 1 & 7 & 17 & 3 \\ 2 & 2 & 4 & 3 \end{pmatrix}$$

Using elementary transformations (elementary row and column operations), find the value of ρ that minimizes the rank of C.