MATH 235/W08: Orthogonal Diagonalization, Symmetric & Complex Matrices, Assignment 8

Hand in questions 1,3,5,7,9,11,13 by 9:30 am on Wednesday April 2, 2008.

Contents

1 Properties of Symmetric/Hermitian/Normal Matrices***

A (complex) *normal matrix* is defined by $A^*A = AA^*$; it has orthogonal eigenvectors. A *skew-Hermitian matrix* is defined by $A^* = -A$.

1. Why is every skew-Hermitian matrix normal?
2. Why is every unitary matrix normal?
3. For what values of a, d is the 2×2 matrix $\begin{pmatrix} a & 1 \\ -1 & d \end{pmatrix}$ normal?

2 More on Hermitian/Unitary Matrices

1. Let A, B be $n \times n$ matrices and suppose $B = A^{-1}A^T$ and B is symmetric. Prove that A^2 is symmetric.
2. Suppose C is a real $n \times n$ matrix such that C is symmetric and $C^2 = C$ and let $D = I_n - 2C$ with I_n denoting the $n \times n$ identity matrix. Prove that D is symmetric and orthogonal.
3. Find all complex 2×2 matrices $A = [a_{ij}]$ which are both unitary and Hermitian, and have $a_{11} = 1/2$.

3 Hermitian, Orthogonal Projections***

Let Z be an $m \times n$ complex matrix such that $Z^*Z = I_n$ where I_n denotes the $n \times n$ identity matrix.

1. Show that $H = ZZ^*$ is Hermitian and satisfies $H^2 = H$.
2. Show that $U = I_n - 2ZZ^*$ is both unitary and Hermitian.

4 Hermitian and Skew-Hermitian Parts

Let A be a complex $n \times n$ matrix.

1. Show that $A = H + K$ for some Hermitian matrix H and some skew-Hermitian matrix K.
2. Show that H and K in part (a) are unique.
3. For H and K defined in part (a), show that $AA^* = A^*A$ if and only if $HK = KH$.

5 Quadratic Forms***

Let A be a real $n \times n$ matrix and let H be a complex $n \times n$ Hermitian matrix.

1. Find a real *symmetric* $n \times n$ matrix B such that the quadratic forms $x^TAx = x^TBx, \forall x \in \mathbb{R}^n$.
2. Verify that $\mathbf{x}^*H\mathbf{x} \in \mathbb{R}$, for all $\mathbf{x} \in \mathbb{C}^n$.
3. Show that if a Hermitian matrix H can be written as $H = A^*A$ for some invertible complex matrix A, then $\mathbf{x}^*H\mathbf{x} > 0$ for all nonzero vectors $\mathbf{x} \in \mathbb{C}^n$.

6 Normal Matrices

Recall that an $n \times n$ complex matrix N is *normal* if $N^*N = NN^*$ where $N^* = \overline{N}^T$. Prove that if N is normal, then $N - cI_n$ is also normal for any complex scalar c. Here, I_n denotes the $n \times n$ identity matrix.

7 Orthogonal Diagonalization***

Consider the real symmetric matrix $A = \begin{bmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{bmatrix}$.

1. Find an orthogonal matrix P and a diagonal matrix D such that $A = PDP^T$.

2. Find a 3×3 real symmetric matrix X such that $X^3 = A$.

8 Eigenspaces

Consider the complex Hermitian matrix $C = \begin{bmatrix} 5 & 2-i & -1+i \\ 2+i & 1 & 3-i \\ -1-i & 3+i & 4 \end{bmatrix}$.

(a) Find the eigenvalues of C and their corresponding eigenspaces. Note that the sum along every row of C is 6.

(b) Find a unitary matrix U and a diagonal matrix D such that $C = UDU^*$.

9 Unitary Diagonalization***

1. A matrix H_s over $\mathbf{C}$ is skew-Hermitian if $H_s^* = -H_s$. Prove that every eigenvalue of a skew-Hermitian matrix H_s has real part zero.

2. Find a unitary diagonalization of the following skew-symmetric matrix
   $A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$.

10 Symmetric Square Root

Find a symmetric matrix B such that $B^2 = \begin{bmatrix} 17 & 16 & -16 \\ 16 & 41 & -32 \\ -16 & -32 & 41 \end{bmatrix}$.

11 Orthogonal Eigenvectors***

This A is *nearly symmetric*. But its eigenvectors are far from orthogonal: $A = \begin{pmatrix} 1 & 10^{-5} \\ 0 & 1+10^{-5} \end{pmatrix}$.
One eigenvector is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. What is another linearly independent eigenvector and what is the angle between the two eigenvectors?

12 Common Eigenpairs

Note that a well known theorem states: $AB = BA$ *implies that* A, B *share the same eigenvectors*. Suppose that A is normal. Therefore, $AA^T = A^T A$ and so A and A^T share the same eigenvectors. But A and A^T always share the same eigenvalues. Therefore they must have the same matrices U, D in a unitary diagonalization. Therefore, $A = A^T$? Where is the paradox?

13 MATLAB***

13.1 Colliding Eigenvalues***

Choose two simple 2×2 symmetric matrices with *different eigenvectors*. Say $A = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ and another nondiagonal symmetric matrix. Graph the two eigenvalues of $A+tB$ as t varies $-8 < t < 8$. Use e.g.

```
t=linspace(-8,8);
l1s=[];
l2s=[];
for i=1:length(t),
   l=eig(A+t(i)*B);
   l1s=[l1s l(1)];
   l2s=[l2s l(2)];
end
plot(t,l1s,'ob')
hold on
plot(t,l2s,'xr')
hold off
```

Note that the eigenvalues appear to be on a collision course, yet at the last minute they turn aside. How close do they come?

13.2 Equation of an Orbit***

The general equation of a conic section in the plane (a parabola, hyperbola, ellipse, or degenerate forms of these) is given by

$$c_1x^2 + c_2xy + c_3y^2 + c_4x + c_5y + c_6 = 0$$

Given five distinct points on the conic, the constants $c_1, \cdots, c_6$ can be determined and will be unique to within a multiplicative constant. To see why this is so, let $(x_i, y_i), i = 1, \cdots, 5$ denote the distinct points. Then, we can form the following system of equations:

$$x^2c_1 + xyc_2 + y^2c_3 + xc_4 + yc_5 + c_6 = 0$$

$$x_1^2c_1 + x_1y_1c_2 + y_1^2c_3 + x_1c_4 + y_1c_5 + c_6 = 0$$
$$x_2^2c_1 + x_2y_2c_2 + y_2^2c_3 + x_2c_4 + y_2c_5 + c_6 = 0$$
$$x_3^2c_1 + x_3y_3c_2 + y_3^2c_3 + x_3c_4 + y_3c_5 + c_6 = 0$$
$$x_4^2c_1 + x_4y_4c_2 + y_4^2c_3 + x_4c_4 + y_4c_5 + c_6 = 0$$
$$x_5^2c_1 + x_5y_5c_2 + y_5^2c_3 + x_5c_4 + y_5c_5 + c_6 = 0$$

This system can be written in the form of a homogeneous linear system of six equations for the six unknowns $c_1, \cdots, c_6$. Because $c_1, \cdots, c_6$ are not all zero, this system has a nontrivial solution.

Now, recall that a homogeneous linear system with as many equations as unknowns has a nontrivial solution if and only if the determinant of the coefficient matrix is zero. Thus, we must have that

$$\begin{vmatrix} x^2 & xy & y^2 & x & y & 1 \\ x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{vmatrix} = 0 \quad (*)$$

Hence, every point (x, y) on the conic must satisfy $(*)$; conversely, it can be shown that every point (x, y) that satisfies $(*)$ lies on the conic. So, $(*)$ represents the equation of the conic.

Use this result to determine the orbit of an asteroid about the sun. Let the sun be positioned at the origin of a Cartesian coordinate system in the plane of the orbit. An astronomer makes five observations of the asteroid at five different times and finds five distinct points $(x_i, y_i), i = 1, \cdots, 5$ along the orbit to be:
$(8.025, 8.310)$, $(10.170, 6.355)$, $(11.202, 3.212)$, $(10.736, 0.375)$, $(9.092, -2.267)$
Here, astronomical units of measurement are used along the axes where 1 astronomical unit = mean earth - to - sun distance (i.e. 150 million kms). With the aid of MATLAB, find the equation of the orbit.