MATH 235/W08 Assignment 4A
Eigenvalues, Eigenvectors, Diagonalization

Hand in questions 2,3,4,5,6,7,12.

Due by 9:30 am on Wed. Feb. 27/08.

1. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ a & 0 & 2 & 0 \\ c & 0 & b & 2 \end{bmatrix}$. Is A diagonalizable?

2. Diagonalize, if possible, the following matrices, i.e. find the eigenvalues and then find the corresponding eigenspaces.

$$A = \begin{bmatrix} 0 & -1 & 3 \\ 5 & 0 & -3 \\ 1 & -1 & 2 \end{bmatrix}, \ B = \begin{bmatrix} -2 & -4i & 4i \\ -4i & 2 & 0 \\ -4i & 0 & 2 \end{bmatrix},$$

$$C = \begin{bmatrix} 4 & 4 & 2 & 3 & -2 \\ 0 & 1 & -2 & -2 & 2 \\ 6 & 12 & 11 & 2 & -4 \\ 9 & 20 & 10 & 10 & -6 \\ 15 & 28 & 14 & 5 & -3 \end{bmatrix}.$$

3. Let $T : \mathbb{P}_2 \to \mathbb{P}_4$ be the transformation that maps a polynomial $p(t)$ into the polynomial $p(t) + t^2 p(t)$.
   (a) Find the image of $p(t) = 2 - t + t^2$.
   (b) Show that T is a linear transformation.
   (c) Find the matrix for T relative to the bases $\{1, t, t^2\}$ and $\{1, t, t^2, t^3, t^4\}$.

4. Suppose that the matrices A, B are similar. Prove that they have the same rank.

5. A 3×3 real matrix A has eigenvalues $\lambda_1 = \lambda_2 = -2, \lambda_3 = 3$ and associated eigenvectors

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \ \mathbf{v_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \ \mathbf{v_3} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

respectively. Find A.

6. Consider the matrix: $A = \begin{bmatrix} 0 & 2 & 0 & 0 \\ k & 0 & 2 & 0 \\ 0 & k & 0 & 2 \\ 0 & 0 & k & 0 \end{bmatrix}$ where $k \in \mathbf{R}$ is a constant.

   (a) Find all values of $k \in \mathbf{R}$ such that A is diagonalizable.

   (b) Find all values of $k \in \mathbf{R}$ such that A is not diagonalizable.

7. Suppose that A is a $n \times n$ real matrix, n is odd. Show that A has at least one real eigenvalue.

8. For $A = \begin{bmatrix} 4 & -6 \\ 3 & -5 \end{bmatrix}$, evaluate A^n for arbitrary positive integer n.

9. Let $A, B \in M_{n\times n}(\mathbf{R})$, we say that A and B are simultaneously diagonalizable (S.D.) if there exists an invertible matrix Q in $M_{n\times n}(\mathbf{R})$ such that

$$Q^{-1}AQ = D_1 \quad \text{and} \quad Q^{-1}BQ = D_2$$

   where D_1 and D_2 are both diagonal matrices in $M_{n\times n}(\mathbf{R})$.

   (a) Suppose that $C \in M_{n\times n}(\mathbf{R})$ and that p is a positive integer. Show that if C is diagonalizable then C and C^p are S.D.

   (b) Suppose that $C \in M_{n\times n}(\mathbf{R})$ and that C is both invertible and diagonalizable. Show that C and C^{-1} are S.D.

   (c) Let $A, B \in M_{n\times n}(\mathbf{R})$ and suppose that A and B are S.D. Show that A and B commute, that is show that $AB = BA$.

10. Let $A \in M_{n\times n}(\mathbf{C})$.

   (a) Show that A and A^T have the same characteristic polynomial and conclude that A and A^T have the same eigenvalues.

   (b) Let λ be an eigenvalue of A with eigenspace E_λ and let $\tilde{E}_\lambda$ be the eigenspace of A^T corresponding to the eigenvalue λ. Note that E_λ and $\tilde{E}_\lambda$ need not be identical. Prove that $\dim(E_\lambda) = \dim(\tilde{E}_\lambda)$.

   (c) Hence, or otherwise, show that if A is diagonalizable then A^T is also diagonalizable.

11. Solve the initial value problem

$$y_1' = 4y_1 + y_3$$

$$y_2' = -2y_1 + y_2$$

$$y_3' = -2y_1 + y_3$$

   where $\mathbf{y}(0) = \begin{bmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

12. (**MATLAB**) **Application of Diagonalization: Linear Recurrences**

A sequence x_0, x_1, ... of numbers is said to be given *recursively* if each number in the sequence is determined by those that precede it. Such sequences often occur in mathematics and science. A linear equation that describes this type of sequence is called a *linear recurrence*.

For example, one of the most famous sequences associated with a linear recurrence equation is the **Fibonacci sequence**. This sequence is generated by the linear recurrence equation

$$x_{n+2} = x_{n+1} + x_n$$

where $x_1 = 1 = x_2$. It follows that

$$x_3 = x_{1+2} = x_2 + x_1 = 1 + 1 = 2,$$

$$x_4 = x_{2+2} = x_3 + x_2 = 2 + 1 = 3,$$

$$x_5 = x_{3+2} = x_4 + x_3 = 3 + 2 = 5,$$

$$x_6 = x_{4+2} = x_5 + x_4 = 5 + 3 = 8,$$

and

$$x_7 = x_{5+2} = x_6 + x_5 = 8 + 5 = 13.$$

In theory, given enough time, we could find the n-th term by first calculating all of the terms preceding it (this would take a very long time if n was large!). However, we can use what we know about diagonalization and eigenvalues to find an explicit formula for the n-th term of this sequence. To see how to do this, consider a general 2nd-order linear recurrence equation:

$$x_{n+2} = bx_{n+1} + ax_n$$

Associate this equation with a matrix A of the form

$$A = \begin{bmatrix} 0 & 1 \\ \mathrm{a} & \mathrm{b} \end{bmatrix}$$

Note that

$$\begin{aligned} A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ \mathrm{a} & \mathrm{b} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} x_2 \\ ax_1 + bx_2 \end{bmatrix} \\ &= \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} \end{aligned}$$

Similarly, we have

$$\begin{aligned} A^2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= A(A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) \\ &= A \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ \text{a} & \text{b} \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} \\ &= \begin{bmatrix} x_3 \\ ax_2 + bx_3 \end{bmatrix} \\ &= \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} \end{aligned}$$

We can continue in this manner to get that for any $n \geq 3$

$$A^{n-2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_{n-1} \\ x_n \end{bmatrix}$$

Suppose that A is diagonalizable with

$$D = P^{-1}AP$$

Then we also have

$$A = PDP^{-1}$$

and hence

$$A^{n-2} = PD^{n-2}P^{-1}$$

Putting this all together would give us that

$$\begin{bmatrix} x_{n-1} \\ x_n \end{bmatrix} = PD^{n-2}P^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

From this equation, we have a means of calculating x_n directly given A, x_1 and x_2.

In the case of the **Fibonacci sequence**, find the associated matrices A, P and D.

Thus, to find the 8th term in the Fibonacci sequence all that is required is one line in MATLAB given by:

```
>> P*D∧6*inv(P)*[1; 1]
```

which returns the vector $[x_7; x_8]$ from which it is found that $x_8 = 21$.

Use Matlab to find the 23rd and 32nd terms in the Fibonacci sequence.