MATH 235/W08 Eigenvalues & Eigenvectors
Assignment 3
due by 9:30 am on Wed. Feb. 6/08.

Notes:
1) The *trace* of an $n \times n$ matrix $A = [a_{ij}]$, denoted $\mathrm{tr}(A)$, is defined to be the sum of the diagonal elements, that is, $\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}$.
2) Our textbook's definition of the characteristic polynomial of an $n \times n$ matrix A, $p_A(\lambda)$, is $p_A(\lambda) := \det(A - \lambda I_n)$. Another commonly used definition is $c_A(\lambda) := \det(\lambda I_n - A)$. These two polynomials are closely related through $p_A(\lambda) = (-1)^n c_A(\lambda)$.

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1. Consider a matrix of the form

$$\begin{pmatrix} 2a-c & a-c & -2a+2c \\ 2a-2b & a & -2a+2b \\ 2a-b-c & a-c & -2a+b+2c \end{pmatrix}$$

   (a) Verify that the two vectors $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$ and $\begin{pmatrix} 1 & 0 & 1 \end{pmatrix}^T$ are eigenvectors and find the corresponding eigenvalues.

   (b) Find a third eigenvalue and a corresponding eigenvector.

2. Let $a, b \in \mathbb{R}, b \neq 0$, and let $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. Find the eigenvalues and eigenvectors of A.

3. For the matrices

$$A = \begin{pmatrix} 1+3i & -4 \\ -2 & 1-3i \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix}:$$

   (i) Find the characteristic polynomials of A and B.

   (ii) Find the eigenvalues of A and B.

   (iii) For each eigenvalue λ of A and B, find a basis for the associated eigenspace.

4. Consider the 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

   (a) Show that the characteristic equation for A can be written as $\lambda^2 - (\text{tr}A)\lambda + \det(A) = 0$.

   (b) Verify the Cayley-Hamilton Theorem for matrix A. (See Problem 7 on page 371 of the text.)

   (c) Prove the inverse formula:

$$A^{-1} = \frac{(\text{tr}A)I - A}{\det(A)} \quad \text{when} \quad \det(A) \neq 0 \,.$$

   (d) Use (c) to obtain the inverse of $A = \begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix}$.

5. **MATLAB**:
   Hand in the output for the following questions. For example, you can use the diary command and hand in the output file, e.g. use diary output-file.txt.

   (a) <u>**EIGSHOW IN MATLAB**</u>
   Try help eigshow in MATLAB. Try using the command eigshow. How can you answer the following three questions by just using eigshow?

   - Which matrices are singular?
   - Which matrices have complex eigenvalues?
   - Which matrices have double (repeated) eigenvalues?

   (b) i. Consider the two matrices

$$A = \begin{pmatrix} -9 & -3 & -16 \\ 13 & 7 & 16 \\ 3 & 3 & 10 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

   And, find matrix C using the command $C = round(3*randn(3))$. Find the characteristic polynomial and the eigenvalues of each of the matrices A, B, C using the commands *poly* and *roots* from MATLAB. Verify your eigenvalues using the command *eig*. (You can use the MATLAB help command, e.g., *help eig*, to get more information on usage.)

   ii. Using the results on the eigenvalues from above and the MATLAB *rref* command, find the eigenspace for each of the eigenvalues of the matrix B, i.e. provide a basis with four decimals accuracy for the eigenspaces. (*eye(3)* is the MATLAB command for the 3×3 identity matrix.)

   (c) <u>**The Power Method and Dominant Eigenvalues**</u>

In practice, the precise eigenvalues are seldom known. Instead, close numerical approximations are used. In many problems, only the eigenvalue having the largest absolute value–sometimes called the *dominant* eigenvalue–is required. An algorithm called the *Power Method* can work well for finding the dominant eigenvalue.

Power Method for Estimating a Strictly Dominant Eigenvalue

i. Select an initial vector x_0 whose largest entry is 1.

ii. For $k = 0, 1, \ldots$

   A. Compute $A\,x_k$.

   B. Let μ_k be an entry in $A\,x_k$ whose absolute value is as large as possible.

   C. Compute $x_{k+1} = (1/\mu_k)\,A\,x_k$. (scale x_{k+1})

iii. For almost all choices of x_0, the sequence $\{\mu_k\}$ approaches the dominant eigenvalue, and the sequence $\{x_k\}$ approaches the corresponding eigenvector.

The following is the MATLAB code that implements the Power Method for a matrix `A` and initial vector `x0`. This algorithm assumes that

i. matrix `A` has a strictly dominant eigenvalue, and

ii. the initial vector `x0` has as its largest entry 1 (in magnitude).

To understand this code, use the matrix

$$A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}$$

with initial vector

$$x0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

In MATLAB, enter

```
>> A = [ 6 5; 1 2]      %Create matrix A
>> x = [ 0; 1]     %Create the initial column vector x0
>> n = 5     %Number of times to execute the loop, in this
case 5
>> format long     % Tell MATLAB to display 15 decimal digits
in the data
```

When the following sequence of commands is executed, the values of `x` can approach an eigenvector for a strictly dominant eigenvalue:

```
>> for j = 1:n     %Execute the following lines n times
y = A*x
[t r] = max(abs(y));
mu = y(r)     %Estimate for the eigenvalue
x = y/y(r)     %Estimate for the eigenvector
end
```

Note that in the line `[t r] = max(abs(y)); mu = y(r)`, `t` is the absolute value of the largest entry in `y` and `r` is the index of that entry. As these commands are repeated, the numbers that appear for `y(r)` are the μ_k that approach the dominant eigenvalue.

The command `format` returns the data display to normal in MATLAB.

```
>> format
```

The MATLAB output from this loop strongly suggests that $\{x_k\}$ approaches $[1, 0.2]$ and μ_k approaches 7. This means that $[1, 0.2]$ is an eigenvector and 7 is the dominant eigenvalue. You can verify this in MATLAB with the following calculation:

```
>> x = [1 ; 0.2]
>> A*x
>> 7*x
```

Indeed, MATLAB shows that $Ax = 7x$ and we have found the dominant eigenvalue.

Using the Power Method loop presented above with the initial guess $x0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $n = 25$, calculate the dominant eigenvalue of matrix A to three decimal place accuracy and its corresponding eigenvector, where

$$A = \begin{pmatrix} 4 & 3 & 1 \\ 1 & -3 & 2 \\ 3 & .2 & 2 \end{pmatrix}.$$

Compare your answer with the result obtained using the command [P, D]=eig(A).