MATH 235/W08 Determinants

Assignment 2

due by 9:30 am on Wed. Jan. 23/08.

Hand in questions 1,2,3,4,5,6,7,8

1. Evaluate the determinants of the following matrices:

   (a)
$$\begin{pmatrix} 1 & i & 1+i \\ 2-i & 3 & 0 \\ -1+i & 3 & 1-i \end{pmatrix}$$

   (b)
$$\begin{pmatrix} 0 & 2 & 3 & \pi \\ 1 & -1.56 & -1.58 & 4.34 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 9 & 102 \end{pmatrix}$$

   (c)
$$\begin{pmatrix} a & b & c \\ 2a+3d & 2b+3e & 2c+3f \\ d & e & f \end{pmatrix}, \quad \text{where } a,b,c,d,e,f \in \mathbb{C}.$$

   (d)
$$\begin{pmatrix} 6 & 0 & -1 & 0 & 0 \\ 9 & 3 & 2 & 3 & 7 \\ 8 & 0 & 3 & 2 & 9 \\ 0 & 0 & 4 & 0 & 0 \\ 5 & 0 & 5 & 0 & 1 \end{pmatrix}$$

2. Find the value of the determinants of the following matrices. Explain your answer in terms of cofactor expansion:

   (a)
$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

(b) The 100×100 matrix A below:

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

The ijth entry of A is 1 if $i + j = 101$; all other entries are 0.

3. Evaluate the following determinants by row-reducing to an upper triangular matrix:

(a)

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 4 & 4 \\ 1 & -1 & 2 & -2 \\ 1 & -1 & 8 & -8 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 6 & 6 & 1 \\ -1 & 0 & 0 & 3 \\ 0 & 2 & 0 & 5 \end{pmatrix}$$

(c)

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

4. Suppose that $A \in \mathcal{M}_{4\times 4}(\mathbb{C})$ i.e. a 4×4 matrix with complex entries. Suppose that the columns of A are given by the 4 vectors v_1, v_2, v_3, v_4 and $\det(A) = 9$. Find:

$$\det \begin{pmatrix} v_1 & v_4 & v_3 & v_2 \end{pmatrix}; \qquad \det \begin{pmatrix} 6v_1 + 2v_4 & v_2 & v_3 & 3v_1 + v_4 \end{pmatrix}$$

5. Find the determinant of the matrix representation (with respect to the standard basis) of the linear operator $T(p(t)) = p(4t - 1)$, where $T : P_2 \to P_2$ i.e. T is a linear operator on the vector space of polynomials of degree 2.

6. Use the determinant of an appropriate matrix to find the area of the triangle defined by $\begin{pmatrix} 3 \\ 7 \end{pmatrix}$ and $\begin{pmatrix} 8 \\ 2 \end{pmatrix}$.

7. Let A be an $n \times n$ matrix with integer entries and with $\det(A) = 1$. Are the entries of A^{-1} integers? Why?

8. **MATLAB:** You will need the two Teaching Code file cofactor.m available from the course webpage or
http://web.mit.edu/18.06/www/Course-Info/Mfiles/cofactor.m

   Use the diary command to save your input/output in MATLAB, e.g. diary inoutassign2.txt will save your output in the file inoutassign2.txt.

   (a) Create a random 5×5 matrix A using the MATLAB randn command. (Use help randn in MATLAB if you need to.)

   (b) Evaluate the determinant using the MATLAB det command. Then exchange rows 3 and 5: B = A(:,[1 2 5 4 3]). Use MATLAB to verify that det B = - det A.

   (c) Now let C be the matrix obtained from A by adding 3 times row 4 to row 1. Again, use MATLAB to verify that now the determinant is unchanged.

   (d) The file cofactor.m calculates the matrix of cofactors of a square matrix. Construct an integer matrix from A using D=round(2*A). Then find the cofactor matrix using COF=cofactor(D). The adjunct matrix is ADJ=COF'. Find the inverse of D using ADJ.

   (e) Generate a random integer vector b to solve the system Dx=b using Cramer's rule. Verify that you have the solution using the \ command in MATLAB.

9. For each of the following, find all values of x for which the matrix is invertible.

   (a) $\begin{bmatrix} \cos x & 1 & -\sin x \\ 0 & 2 & 0 \\ \sin x & 3 & \cos x \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & 1 & x \\ 1 & x & x \\ x & x & x \end{bmatrix}$

10. Theorem 9, p. 205, of Lay interprets det$A$ in terms of area and volume. This generalizes to
Theorem: The $k-$volume of a $k-$parallelepiped defined by vectors $v_1, \ldots, v_k$ in $\mathbb{R}^n$ is $\sqrt{\det(A^T A)}$ where A is the $n \times k$ matrix whose column vectors are $v_1, \ldots, v_k$. When $k = n$, this reduced to $|\det A|$.

    Note: 2-volume = area; 2-parallelepiped= parallelogram, etc. Apply this Theorem to answer the following problems:

    (a) Find the area of the parallelogram defined by the vectors
    $$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

(b) Find the volume of the parallelpiped defined by the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

11. (a) Prove that $\det \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} = (\det A)(\det D)$, if A and D are square blocks not necessarily of the same size. (Hint: Expand along the last row of D and use induction.)

(b) Let a $2n \times 2n$ matrix M be in the form $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where each block is an $n \times n$ matrix. Suppose that A is invertible and that $AC = CA$. Prove that $\det M = \det(AD - CB)$. (Hint: Post-multiply M by the triangular matrix $\begin{bmatrix} I_n & -A^{-1}B \\ 0 & I_n \end{bmatrix}$, whose determinant has value 1.)