SYMMETRY IN SEMIDEFINITE PROGRAMS

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ABSTRACT. This paper is a tutorial in a general and explicit procedure to simplify semidefinite programs which are invariant under the action of a symmetry group. The procedure is based on basic notions of representation theory of finite groups. As an example we derive the block diagonalization of the Terwilliger algebra of the binary Hamming scheme in this framework. Here its connection to the orthogonal Hahn and Krawtchouk polynomials becomes visible.

1. INTRODUCTION

A *(complex) semidefinite program* is an optimization problem of the form

$$\max\{\langle C, Y\rangle : \langle A_i, Y\rangle = b_i,\ i = 1, \ldots, n, \text{ and } Y \succeq 0\}, \tag{1}$$

where $A_i \in \mathbb{C}^{X\times X}$, and $C \in \mathbb{C}^{X\times X}$ are given Hermitian matrices whose rows and columns are indexed by a finite set X, $(b_1, \ldots, b_n)^t \in \mathbb{R}^n$ is a given vector and $Y \in \mathbb{C}^{X\times X}$ is a variable Hermitian matrix and where "$Y \succeq 0$" means that Y is positive semidefinite. Here $\langle C, Y\rangle = \mathrm{trace}(CY)$ denotes the trace product between symmetric matrices.

Semidefinite programming is an extension of linear programming and has a wide range of applications: combinatorial optimization and control theory are the most famous ones. Although semidefinite programming has an enormous expressive power in formulating convex optimization problems it has a few practical drawbacks: Highly robust and highly efficient solvers, unlike their counterparts for solving linear programs, are currently not available. So it is crucial to exploit the problems' structure to be able to perform computations.

In the last years many results were obtained if the problem under consideration has symmetry. This was done for a variety of problems and applications: interior point algorithms (Kanno, Ohsaki, Murota, Katoh [16] and de Klerk, Pasechnik [5]), polynomial optimization (Parrilo, Gatermann [10] and Jansson, Lasserre, Riener, Theobald [14]), truss topology optimization (Bai, de Klerk, Pasechnik, Sotirov [3]), quadratic assignment (de Klerk, Sotirov [7]), fast mixing Markov chains on graphs (Boyd, Diaconis, Xiao [4]), graph coloring (Gvozdenović, Laurent [13]),

Date: July 30, 2008.

1991 *Mathematics Subject Classification.* 90C22, 33C90.

Key words and phrases. semidefinite programming, block diagonalization, Terwilliger algebra, binary Hamming scheme, Hahn polynomials.

The author was supported by the Netherlands Organization for Scientific Research under grant NWO 639.032.203 and by the Deutsche Forschungsgemeinschaft (DFG) under grant SCHU 1503/4-2.

crossing numbers for complete binary graphs (de Klerk, Pasechnik, Schrijver [6]) and coding theory (Schrijver [20], Gijswijt, Schrijver, Tanaka [11] and Laurent [18]).

In all these applications the underlying principles are similar: one simplifies the original semidefinite program which is invariant under a group action by applying an algebra isomorphism mapping a "large" matrix algebra to a "small" matrix algebra. Then it is sufficient to solve the semidefinite program using the smaller matrices. The existence of an appropriate algebra isomorphism is a classical fact from Artin-Wedderburn theory. However, in the above mentioned papers the explicit determination of an appropriate isomorphism is rather mysterious. The aim of this paper is to give an algorithmic way to do this which also is well-suited for symbolic calculations by hand.

The paper is structured as follows: Section 2 recalls basic definitions and shows how the Artin-Wedderburn theorem stated in (4) can be applied to simplify a semidefinite program invariant under a group action. In Section 3 we construct an explicit algebra isomorphism. In Section 4 we apply this to the Terwilliger algebra of the binary Hamming scheme.

This paper is of expository nature and probably few of the results are new. On the other hand a tutorial of how to use symmetry in semidefinite programming is not readily available. Furthermore our treatment of the Terwilliger algebra for binary codes provides an alternative point of view which emphasizes the action of the symmetric group. Schrijver [20] treated the Terwilliger algebra with elementary combinatorial and linear algebraic arguments. Our derivation has the advantage that it gives an interpretation for the matrix entries in terms of Hahn polynomials. In a similar way one can derive the block diagonalization of the Terwilliger algebra for nonbinary codes which was computed by Gijswijt, Schrijver, Tanaka [11]. Here products of Hahn and Krawtchouk polynomials occur.

2. Background and notation

In this section we present the basic framework for simplifying a semidefinite program invariant under a group action.

Let G be a finite group which acts on a finite set X by $(a, x) \mapsto ax$ with $a \in G$ and $x \in X$. This group action extends to an action on pairs $(x, y) \in X \times X$ by $(a, (x, y)) \mapsto (ax, ay)$. In this way it extends to square matrices whose rows and columns are indexed by X: for an $X \times X$-matrix M we have $aM(x, y) = M(ax, ay)$. Here $M(x, y)$ denotes the entry of M at position (x, y). A matrix M is called *invariant under* G if $M = aM$ for all $a \in G$.

A Hermitian matrix $Y \in \mathbb{C}^{X \times X}$ is called a *feasible solution* of (1) if it fulfills the conditions $\langle A_i, Y\rangle = b_i$ and $Y \succeq 0$. It is called an *optimal solution* if it is feasible and if for all other feasible solutions Y' we have $\langle C, Y\rangle \geq \langle C, Y'\rangle$. In the following we assume that the semidefinite program (1) has an optimal solution.

We say that the semidefinite program (1) is *invariant under* G if for every feasible solution Y and for every $a \in G$ the matrix aY is again a feasible solution and if it is satisfies $\langle C, aY\rangle = \langle C, Y\rangle$ for all $a \in G$. Because of the convexity of

(1), one can find an optimal solution of (1) in the subspace $\mathcal{B}$ of matrices which are invariant under G. In fact, if Y is an optimal solution of (1), so is its *group average* $\frac{1}{|G|}\sum_{a\in G} aY$. Hence, (1) is equivalent to

$$\max\{\langle C, Y\rangle : \langle A_i, Y\rangle = b_i,\ i = 1,\dots,n,\ Y \succeq 0, \text{ and } Y \in \mathcal{B}\}. \tag{2}$$

The set $X\times X$ can be decomposed into the orbits $R_1,\dots,R_N$ by the action of G. For every $r \in \{1,\dots,N\}$ we define the matrix $B_r \in \{0,1\}^{X\times X}$ by $B_r(x,y) = 1$ if $(x,y) \in R_r$ and $B_r(x,y) = 0$ otherwise. Then $B_1,\dots,B_N$ forms a basis of $\mathcal{B}$. We call $B_1,\dots,B_N$ the *canonical basis* of $\mathcal{B}$. If $(x,y) \in R_r$ we also write $B_{[x,y]}$ instead of B_r. Note that $B_{[y,x]}$ is the transpose of the matrix $B_{[x,y]}$.

So the first step to simplify a semidefinite program which is invariant under a group is as follows:

If the semidefinite program (1) *is invariant under G, then* (1) *is equivalent to*

$$\max\Big\{c_1y_1 + \cdots + c_Ny_N \ :\ \begin{array}{l} y_1,\dots,y_N \in \mathbb{C},\\ a_{i1}y_1 + \cdots + a_{iN}y_N = b_i,\ i = 1,\dots,n,\\ y_j = \overline{y_k} \text{ if } B_j = (B_k)^t,\\ y_1B_1 + \cdots + y_NB_N \succeq 0\Big\},\end{array} \tag{3}$$

where $c_r = \langle C, B_r\rangle$, and $a_{ir} = \langle A_i, B_r\rangle$.

The following obvious property is crucial for the next step of simplifying (3): The subspace $\mathcal{B}$ is closed under matrix multiplication. So $\mathcal{B}$ is a (semisimple) algebra over the complex numbers. The Artin-Wedderburn theory (cf. [17, Chapter 1]) gives:

There are numbers d, and $m_1,\dots,m_d$ so that there is an algebra isomorphism

$$\varphi : \mathcal{B} \to \bigoplus_{k=1}^{d} \mathbb{C}^{m_k\times m_k}. \tag{4}$$

This applied to (3) gives the final step of simplifying (1):

If the semidefinite program (1) *is invariant under G, then* (1) *is equivalent to*

$$\max\Big\{c_1y_1 + \cdots + c_Ny_N \ :\ \begin{array}{l} y_1,\dots,y_N \in \mathbb{C},\\ a_{i1}y_1 + \cdots + a_{iN}y_N = b_i,\ i = 1,\dots,n,\\ y_j = \overline{y_k} \text{ if } B_j = (B_k)^t,\\ y_1\varphi(B_1) + \cdots + y_N\varphi(B_N) \succeq 0\Big\}.\end{array} \tag{5}$$

Notice that since φ is an algebra isomorphism between matrix algebras with unity, φ preserves eigenvalues and hence positive semidefiniteness. In accordance to the literature, applying φ to a semidefinite program is called *block diagonalization*.

The advantage of (5) is that instead of dealing with matrices of size $|X|\times|X|$ one has to deal with block diagonal matrices with d block matrices of size $m_1,\dots,m_d$, respectively. In many applications the sum $m_1 + \cdots + m_d$ is much smaller than

$|X|$ and in particular many practical solvers take advantage of the block structure to speed up the numerical calculations.

3. Determining a block diagonalization

In this section we give an explicit construction of an algebra isomorphism φ. It has two main features: One can turn the construction into an algorithm as we show at the end of this section, and one can use it for symbolic calculations by hand as we demonstrate in Section 4.

3.1. **Construction.** We begin with some basic notions from representation theory of finite groups. Consider the complex vector space $\mathbb{C}^X$ of vectors indexed by X with inner product $(f, g) = \frac{1}{|X|}\sum_{x \in X} f(x)\overline{g(x)}$. The group G acts on $\mathbb{C}^X$ by $af(x) = f(a^{-1}x)$. Note that the inner product on $\mathbb{C}^X$ is invariant under the group action: For all $f, g \in \mathbb{C}^X$ and all $a \in G$ we have $(af, ag) = (f, g)$. A subspace $H \subseteq \mathbb{C}^X$ is called a *G-space* if $GH \subseteq H$ where $GH = \{af : f \in H, a \in G\}$. It is called *irreducible* if the only proper subspace $H' \subseteq H$ with $GH' \subseteq H'$ is $\{0\}$. Two G-spaces H and H' are called *equivalent* if there is a *G-isometry* $\phi : H \to H'$, i.e. a linear isomorphism with $\phi(af) = a\phi(f)$ for all $f \in H$ and $a \in G$ and $(\phi(f), \phi(g)) = (f, g)$ for all $f, g \in H$.

By Maschke's theorem (cf. [12, Theorem 2.4.1]) one can decompose $\mathbb{C}^X$ orthogonally into irreducible G-spaces:

$$\mathbb{C}^X = (H_{1,1} \perp \ldots \perp H_{1,m_1}) \perp \ldots \perp (H_{d,1} \perp \ldots \perp H_{d,m_d}), \tag{6}$$

where $H_{k,i}$ with $k = 1, \ldots, d$ and $i = 1, \ldots, m_k$ is an irreducible G-space of dimension h_k and where $H_{k,i}$ and $H_{k',i'}$ are equivalent if and only if $k = k'$.

Let $\mathcal{A}$ be the subalgebra of $\mathbb{C}^{X \times X}$ which is generated by the permutation matrices $P_a \in \mathbb{C}^{X \times X}$ with $a \in G$ where

$$P_a(x, y) = \begin{cases} 1 & \text{if } a^{-1}x = y, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Because of (6) the algebra $\mathcal{A}$ decomposes as a complex vector space in the following way

$$\mathcal{A} \cong \bigoplus_{k=1}^{d} \mathbb{C}^{h_k \times h_k} \otimes I_{m_k}. \tag{8}$$

Recall that by $\mathcal{B}$ we denote the matrices in $\mathbb{C}^{X \times X}$ which are invariant under the group action of G. In other words, it is the *commutant* of $\mathcal{A}$:

$$\mathcal{B} = \mathrm{Comm}(\mathcal{A}) = \{B \in \mathbb{C}^{X \times X} : BA = AB \text{ for all } A \in \mathcal{A}\}.$$

The double commutant theorem [12, Theorem 3.3.7] gives the following decomposition of $\mathcal{B}$ as a complex vector space:

$$\mathcal{B} \cong \bigoplus_{k=1}^{d} I_{h_k} \otimes \mathbb{C}^{m_k \times m_k}. \tag{9}$$

Now we construct an explicit algebra isomorphism between the commutant algebra $\mathcal{B}$ and matrix algebra $\bigoplus_{k=1}^{d} \mathbb{C}^{m_k \times m_k}$.

Let $e_{k,1,l}$ with $l = 1, \ldots, h_k$ be an orthonormal basis of the space $H_{k,1}$. Choose G-isometries $\phi_{k,i} : H_{k,1} \to H_{k,i}$. Then, $e_{k,i,l} = \phi_{k,i}(e_{k,1,l})$ is an orthonormal basis of $H_{k,i}$. Define the matrix $E_{k,i,j} \in \mathbb{C}^{X \times X}$ with $i, j = 1, \ldots, m_k$ by

$$E_{k,i,j}(x, y) = \frac{1}{|X|} \sum_{l=1}^{h_k} e_{k,i,l}(x) \overline{e_{k,j,l}(y)}.$$

The definition of these matrices depend on the choice of the orthonormal basis, on the chosen G-isometries and on the chosen decomposition (6). The following proposition shows the effect of different choices.

Proposition 3.1. *By $E_k(x, y)$ we denote the $m_k \times m_k$ matrix $(E_{k,i,j}(x, y))_{i,j}$.*

(a) The matrix entries $E_{k,i,j}(x, y)$ do not depend on the choice of the orthonormal basis of $H_{k,1}$.

(b) The change of $\phi_{k,i}$ to $\alpha\phi_{k,i}$ with $\alpha \in \mathbb{C}$, $|\alpha| = 1$, simultaneously changes the i-th row and i-th column in the matrix $E_k(x, y)$ by a multiplication with α and $\overline{\alpha}$, respectively.

(c) The choice of another decomposition of $H_{k,1} \perp \ldots \perp H_{k,m_k}$ as a sum of m_k orthogonal, irreducible G-spaces changes $E_k(x, y)$ to $UE_k(x, y)\overline{U}^t$ for some unitary matrix $U \in \mathrm{U}(\mathbb{C}^{m_k})$.

Proof. This was proved in [2, Theorem 3.1] with the only difference that there only the real case was considered. The complex case follows mutatis mutandis. □

The following theorem shows that the map

$$\varphi : \mathcal{B} \to \bigoplus_{k=1}^{d} \mathbb{C}^{m_k \times m_k} \tag{10}$$

mapping $E_{k,i,j}$ to the elementary matrix with the only non-zero entry 1 at position (i, j) in the k-th summand $\mathbb{C}^{m_k \times m_k}$ of the direct sum is an algebra isomorphism.

Theorem 3.2. *The matrices $E_{k,i,j}$ form a basis of $\mathcal{B}$ satisfying the equation*

$$E_{k,i,j} E_{k',i',j'} = \delta_{k,k'} \delta_{j,i'} E_{k,i,j'}, \tag{11}$$

where δ denotes Kronecker's delta.

Proof. The multiplication formula (11) is a direct consequence of the orthonormality of the vectors $e_{k,i,l}$. That $E_{k,i,j}$ is an element of $\mathcal{B}$ follows from [2, Theorem 3.1 (c)]. From (11) it follows that the matrices $E_{k,i,j}$ are linearly independent, they span a vector space of dimension $\sum_{k=1}^{d} m_k^2$. Hence, by (9), they form a basis of the commutant $\mathcal{B}$. □

Now the expansion of the canonical basis B_r, with $r = 1, \ldots, N$, in the basis $E_{k,i,j}$ with coefficients $p_r(k,i,j)$

$$B_r = \sum_{k=1}^{d} \sum_{i,j=1}^{m_k} p_r(k,i,j) E_{k,i,j}. \tag{12}$$

yields

$$\varphi(B_r) = \sum_{k=1}^{d} \sum_{i,j=1}^{m_k} p_r(k,i,j) \varphi(E_{k,i,j}).$$

3.2. **Orthogonality relation.** For the computation of the coefficients $p_r(k,i,j)$ the following orthogonality relation is often helpful.

If we expand the basis $|X|E_{k,i,j}$ in the canonical basis B_r we get a relation which after normalization is inverse to (12)

$$|X|E_{k,i,j} = \sum_{r=1}^{N} q_{k,i,j}(r) B_r. \tag{13}$$

So we have an orthogonality relation between the $q_{k,i,j}$:

Lemma 3.3. *Let* $v_r = |\{(x,y) \in X \times X : (x,y) \in R_r\}|$. *Then,*

$$\sum_{r=1}^{N} v_r q_{k,i,j}(r) \overline{q_{k',i',j'}(r)} = \delta_{k,k'} \delta_{j,j'} \delta_{i,i'} |X|^2 h_k. \tag{14}$$

Proof. Consider the sum $\sum_{x \in X} E_{k,i,j} E_{k',j',i'}(x,x)$.

On the one hand it is equal to

$$\sum_{x \in X} \delta_{k,k'} \delta_{j,j'} E_{k,i,i'}(x,x) = \delta_{k,k'} \delta_{j,j'} \operatorname{trace} E_{k,i,i'},$$

and

$$\operatorname{trace} E_{k,i,i'} = \sum_{l=1}^{h_k} (e_{k,i,l}, e_{k,i',l}) = \delta_{i,i'} h_k,$$

On the other hand it is

$$\sum_{x \in X} \sum_{y \in X} E_{k,i,j}(x,y) E_{k',j',i'}(y,x) = \frac{1}{|X|^2} \sum_{r=1}^{N} v_r q_{k,i,j}(r) \overline{q_{k',i',j'}(r)},$$

where we used the fact $E_{k',j',i'}(y,x) = \overline{E_{k',i',j'}(x,y)}$ which follows from the definition. □

The orthogonality relation gives a direct way to compute $p_r(k,i,j)$ once $q_{k,i,j}(r)$ is known: We have

$$p_r(k,i,j) = \frac{v_r \overline{q_{k,i,j}(r)}}{|X| h_k}, \tag{15}$$

which follows by Lemma 3.3 and by (12) and (13) because of

$$\sum_{r=1}^{N} p_r(k,i,j) q_{k',i',j'}(r) = |X| \delta_{k,k'} \delta_{i,i'} \delta_{j,j'}.$$

3.3. **Algorithmic issues.** We conclude this section by reviewing algorithmic issues for computing φ. To calculate the isomorphism one has to perform the following steps:

(1) Compute the orthogonal decomposition (6) of $\mathbb{C}^X$ into pairwise orthogonal, irreducible G-spaces $H_{k,i}$.
(2) For every irreducible G-space $H_{k,1}$ determine an orthonormal basis.
(3) Find G-isometries $\phi_{k,i} : H_{k,1} \to H_{k,i}$.
(4) Express the basis B_r in the basis $E_{k,i,j}$.

Only the first step requires an algorithm which is not classical. Here one can use an algorithm of Babai and Rónyai [1]. It is a randomized algorithm running in expected polynomial time for computing the orthogonal decomposition (6). It requires the permutation matrices P_a given in (7) as input, where a runs through a (favorably small) generating set of G. The other steps can be carried out using Gram-Schmidt orthonormalization and solving systems of linear equations.

4. Block diagonalization of the Terwilliger algebra

The symmetric group S_n acts on the set $X = \{0,1\}^n$ of binary vectors with length n by $\sigma(x_1,\ldots,x_n) = (x_{\sigma(1)},\ldots,x_{\sigma(n)})$, i.e. by permuting coordinates. In [20] Schrijver determined the block diagonalization of the algebra $\mathcal{B}$ of $X \times X$-matrices invariant under this group action. The algebra $\mathcal{B}$ is called the *Terwilliger algebra of the binary Hamming scheme*. Now we shall derive a block diagonalization in the framework of the previous section. In this case it is possible to work over the real numbers only because all irreducible representations of the symmetric group are real.

Under the group action the set X splits into $n+1$ orbits $X_0,\ldots,X_n$ where X_m contains the elements of $\{0,1\}^n$ having Hamming weight m, i.e. elements which one can get from the binary vector $1^m 0^{n-m}$ by permuting coordinates. So we have the orthogonal decomposition of the S_n-space $\mathbb{R}^X$ into

$$\mathbb{R}^X = \mathbb{R}^{X_0} \perp \ldots \perp \mathbb{R}^{X_n}.$$

It is a classical fact (cf. [8, Theorem 2.10]) that the S_n-space $\mathbb{R}^{X_m}$ decomposes further into

$$\mathbb{R}^{X_m} = \begin{cases} H_{0,m} \perp \ldots \perp H_{m,m}, & \text{when } 0 \leq m \leq \lfloor n/2 \rfloor, \\ H_{0,m} \perp \ldots \perp H_{n-m,m}, & \text{otherwise.} \end{cases}$$

where $H_{k,m}$ are irreducible S_n-spaces which correspond to the irreducible representation of S_n given by the partition $(n-k,k)$ (cf. [19, Chapter 2]). Its dimension is $h_k = \binom{n}{k} - \binom{n}{k-1}$.

Thus, the matrices $E_{k,i,j}$, with $k = 0,\ldots,\lfloor n/2 \rfloor$, which correspond to the isotypic component $H_{k,k} \perp \ldots \perp H_{k,n-k}$ of $\mathbb{R}^X$ of type $(n-k,k)$ are conveniently indexed by $i, j = k, \ldots, n-k$. Since $E_{k,j,i}$ is the transpose of $E_{k,i,j}$ we only need to consider the case $k \leq i \leq j \leq n-k$.

To determine $E_{k,i,j}(x,y)$ we rely on the papers [8] and [9] of Dunkl. We recall the facts and notation which we will need from them. Let $T_k : S_n \to \mathrm{O}(\mathbb{R}^{h_k})$ be an orthogonal, irreducible representation of S_n given by the partition $(n-k,k)$. By H, K we denote the subgroups $H = S_j \times S_{n-j}$ and $K = S_i \times S_{n-i}$ of S_n. Let $V_k \subseteq \mathbb{R}^{S_n}$ be the vector space spanned by the function $(T_k)_{rs}$, with $1 \leq r, s \leq h_k$, which are the matrix entries of T_k: $(T_k)_{rs}(\pi) = [T_k(\pi)]_{rs}$. A function $f \in V_k$ is called *H-K-invariant* if $f(\sigma\pi\tau) = f(\pi)$ for all $\sigma \in H$, $\pi \in S_n$, $\tau \in K$. In [8, §4] and [9, §4] Dunkl computed the H-K-invariant functions of V_k. These are all real multiples of

$$\psi_{k,H-K}(\pi) = \frac{(-j)_k(i-n)_k}{(-i)_k(j-n)_k} Q_k(v(\pi); -(n-i)-1, -i-1, j),$$

where $(a)_0 = 1$, $(a)_k = a(a+1)\ldots(a+k-1)$, and where,

$$Q_k(x; -a-1, -b-1, m) = \frac{1}{\binom{m}{k}} \sum_{j=0}^{k} (-1)^j \frac{\binom{b-k+j}{j}}{\binom{a}{j}} \binom{m-x}{k-j} \binom{x}{j},$$

are *Hahn polynomials* (for integers m, a, b with $a \geq m, b \geq m \geq 0$), and where

$$v(\pi) = i - |\pi\{1, \ldots, i\} \cap \{1, \ldots, j\}|.$$

The polynomials $Q_k(x) = Q_k(x; -a-1, -b-1, m)$ are the orthogonal polynomials for the weight function $\binom{a}{x}\binom{b}{m-x}$, $x = 0, 1, \ldots, m$, normalized by $Q_k(0) = 1$. For more information about Hahn polynomials we refer to [15].

We will need the square of the norm of $\psi_{k,H-K}$ which is given in [9, before Proposition 2.7]:

$$(\psi_{k,H-K}, \psi_{k,H-K}) = \frac{\psi_{k,H-K}(\mathrm{id})}{h_k} = \frac{(-j)_k(i-n)_k}{(-i)_k(j-n)_k h_k}.$$

Let $e_{k,i,1}, \ldots, e_{k,i,h_k}$ be an orthonormal basis of $H_{k,i}$. We get an orthogonal, irreducible representation $T_{k,i} : S_n \to \mathrm{O}(\mathbb{R}^{h_k})$ by

$$\pi(e_{k,i,l}) = \sum_{l'=1}^{h_k} [T_{k,i}(\pi)]_{l',l} e_{k,i,l'}.$$

Consider the function

$$z_{k,i,j}(\pi) = E_{k,i,j}(\pi(1^i 0^{n-i}), 1^j 0^{n-j}).$$

This is an H-K-invariant function because $E_{k,i,j} \in \mathcal{B}$. It lies in V_k because vector spaces spanned by matrix entries of two equivalent irreducible representations coincide. Thus, $z_{k,i,j}$ is a real multiple of $\psi_{k,H-K}$. By computing the squared norm of $z_{k,i,j}$ we determine this multiple up to sign:

$$\begin{aligned}(z_{k,i,j}, z_{k,i,j}) &= \frac{1}{n!}\sum_{\pi \in S_n} z_{k,i,j}(\pi) z_{k,i,j}(\pi) \\ &= \frac{1}{\binom{n}{i}2^n}\sum_{l=1}^{h_k}(e_{k,i,l}(1^j 0^{n-j}))^2 \\ &= \frac{1}{\binom{n}{i}} E_{k,j,j}(1^j 0^{n-j}, 1^j 0^{n-j}).\end{aligned}$$

Here we used that $e_{k,i,l}$ is an orthonormal basis of $H_{k,i}$ where the inner product is $(f,g) = \frac{1}{2^n}\sum_{x\in X_i} f(x)g(x)$.

All diagonal entries belonging to $X_j \times X_j$ of $E_{k,j,j}$ coincide and all others are zero, so $\binom{n}{j}E_{k,j,j}(1^j 0^{n-j}, 1^j 0^{n-j})$ is the trace of $E_{k,j,j}$ which equals its rank h_k. Hence, $(z_{k,i,j}, z_{k,i,j}) = h_k(\binom{n}{i}\binom{n}{j})^{-1}$. So we have determined $E_{k,i,j}$ up to sign. To adjust the signs it is enough to ensure that the multiplication formula (11) is satisfied.

So putting it together, we have proved the following theorem.

Theorem 4.1. *For $x, y \in X$ define $v(x,y) = |\{l \in \{1,\ldots,n\} : x_l = 1, y_l = 0\}|$. For $k = 0,\ldots,\lfloor n/2\rfloor$ and $i, j = k,\ldots,n-k$ with $i \le j$ we have*

$$\begin{aligned}E_{k,i,j}(x,y) = \frac{h_k}{(\binom{n}{i}\binom{n}{j})^{1/2}}\left(\frac{(-j)_k(i-n)_k}{(-i)_k(j-n)_k}\right)^{-\frac{1}{2}}\cdot \\ Q_k(v(x,y); -(n-i)-1, -i-1, j),\end{aligned}$$

when $x \in X_i$, $y \in X_j$. In the case $x \notin X_i$ or $y \notin X_j$ we have $E_{k,i,j}(x,y) = 0$. Furthermore, $E_{k,j,i} = (E_{k,i,j})^t$.

Finally, to find the desired algebra isomorphism (4) we determine the values of $p_r(k,i,j)$ by formula (15). We represent the orbits $R_1,\ldots,R_N$ by triples (r,s,d): Two pairs $(x,y),(x',y') \in X\times X$ are equivalent whenever $x, x' \in X_r$, $y, y' \in X_s$, and $v(x,y) = v(x',y') = d$. Then,

$$p_{r,s,d}(k,i,j) = \frac{v_{r,s,d}E_{k,i,j}(x,y)}{h_k},$$

where

$$v_{r,s,d} = \binom{n}{d}\binom{n-d}{r-d}\binom{n-r}{s-s+d}.$$

Remark 4.2. *In a similar way one can give an interpretation of the block diagonalization of the Terwilliger algebra for nonbinary codes which was computed in* [11]*. Using* [8, Theorem 4.2] *one can show the matrix entries are, up to scaling factors, products of Hahn polynomials and Krawtchouk polynomials.*

ACKNOWLEDGEMENTS

We thank Christine Bachoc and the anonymous referee for very helpful remarks and suggestions.

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