# Copositive and Semidefinite Relaxations of the Quadratic Assignment Problem †

Janez Povh ‡ Franz Rendl §

October 24, 2006

**Abstract**

Semidefinite relaxations of the quadratic assignment problem (QAP) have recently turned out to provide good approximations to the optimal value of QAP. We take a systematic look at various conic relaxations of QAP. We first show that QAP can equivalently be formulated as a linear program over the cone of completely positive matrices. Since it is hard to optimize over this cone, we also look at tractable approximations and compare with several relaxations from the literature. We show that several of the well-studied models are in fact equivalent. It is still a challenging task to solve the strongest of these models to reasonable accuracy on instances of moderate size. We also provide a new relaxation, which gives strong lower bounds and is easy to compute.

**Key words**: quadratic assignment problem, copositive programming, semidefinite relaxations, lift-and-project relaxations.

**AMS Subject Classification (2000)**: 90C10, 90C20, 90C22, 90C27.

## 1 Introduction

The quadratic assignment problem (QAP) is a standard problem in location theory and is very famous because of its hardness. Koopmans and Beckmann [11] introduced it in 1957 in the following form:

$$\text{(QAP)} \qquad OPT_{QAP} \;=\; \min\,\{\sum_{i,j} a_{ij} b_{\pi(i)\pi(j)} + \sum_i c_{i,\pi(i)} \colon \pi \text{ a permutation}\},$$

where $A, B, C$ are $n \times n$ matrices. We make the standard assumption that $A$ and $B$ are symmetric. Recent surveys about QAP are given for instance in [5, 19], and most recently in [13].

We may represent each permutation $\pi$ by permutation matrix $X \in \{0,1\}^{n\times n}$, defined by $x_{ij} = 1 \iff \pi(i) = j$. If we denote the set of all permutation matrices by $\Pi$, then we may formulate QAP as follows

$$\text{(QAP)} \qquad OPT_{QAP} \;=\; \min\,\{\langle X,\, AXB + C\rangle \colon X \in \Pi\}.$$

†Partial support by Marie Curie Research Training Network MCRTN-CT-2003-504438 (ADONET) is gratefully acknowledged.

‡University in Maribor, Faculty of logistics, Mariborska cesta 3, Celje, Slovenia (e-mail: janez.povh@uni-mb.si).

§Alpe-Adria University Klagenfurt, Institute for Mathematics, Universitätsstraße 65-67, 9020 Klagenfurt, Austria (e-mail: franz.rendl@uni-klu.ac.at).

QAP is known to be very hard from a theoretical and practical point of view. Problems of size $n \geq 25$ are currently still considered as difficult. Sahni and Gonzales [21] showed that even finding an $\varepsilon$-approximate solution for QAP is NP-hard. Solving QAP in practice is usually based on the Branch and Bound (B&B) algorithm. The performance of B&B algorithms depends on the computational quality and efficiency of lower bounds (see [1] for a summary of recent advances in the solution of QAP by B&B). The study of lower bounds for QAP is therefore very important for the development of B&B algorithms.

The most recent and promising trends of research for the bounding methods for QAP are based on semidefinite programming. Zhao et al., Sotirov and Rendl [22, 20, 24] lifted the problem from vector space $\mathbb{R}^{n\times n}$ to $\mathcal{S}^+_{n^2+1}$ and formulated several SDP which give increasingly tight lower bounds for QAP. They used interior point methods [24] and the bundle method [20] to solve the SDP. The computational results show that these lower bounds are among the strongest but also the most expensive to compute (practical upper bound for the dimension of QAP is $n = 35$).

Recently Burer and Vandenbussche [3] applied the lift-and-project technique, introduced by Lovász and Schrijver [14] to QAP. They used the Augmented Lagrangian method to solve the SDP problem and this way obtained lower bounds for QAP, which are somewhat tighter than the bounds from [20], but the practical upper bound for solving these SDP remains $n = 35$. Adams et al. studied similar technique, named by Reformulation-Linearization Technique, and presented strong lower bounds, based on linear programming,

Our contribution to QAP and computing lower bounds for QAP consists of the following results:

- In Section 2 we show that solving QAP amounts to solving a linear program over the cone of completely positive matrices of order $n^2$. This linear program is actually the conic dual of the Lagrangian dual of the QAP, if we rewrite QAP as a quadratically constrained quadratic problem. This does not make the problem tractable since optimization over the cone of completely positive matrices is intractable, but this result shows new possibilities how to solve QAP approximately.

- In Section 3 we consider the SDP relaxations of QAP, obtained from the copositive representation of QAP from Section 2. We suggest two new SDP models for QAP, denoted by $QAP_{ZKRW1}$ and $QAP_{AW1}$, which both follow from the copositive representation of QAP. The relaxation $QAP_{AW1}$ is a simple improvement of the Anstreicher- Wolkowicz relaxation [2] and can be computed efficiently.

- After having described various previously published relaxations in the sections 4 and 5, we compare these relaxations in section 6. We show that the strongest model $QAP_{K_0^*}$ introduced in the present paper is equivalent to the strongest relaxations from [3, 20, 24]. We also show that $QAP_{ZKRW1}$ is in fact equivalent to the model $QAP_{R_2}$ from [20, 24].

### 1.1 Notation

We denote the $i$th standard unit vector by $e_i$ and when we index components by $0, 1, \ldots, n$, then $e_0$ is the first unit vector. The vector of all ones is $u_n \in \mathbb{R}^n$ (or $u$ if the dimension $n$ is obvious). The square matrix of all ones is $J_n$ (or $J$), the identity matrix is $I$ and $E_{ij} = e_i e_j^T$.

In this paper we consider the following sets of matrices:

- The vector space of real symmetric $n \times n$ matrices: $\mathcal{S}_n = \{X \in \mathbb{R}^{n\times n} : X = X^T\}$,

- the cone of $n \times n$ symmetric nonnegative matrices: $\mathcal{N}_n = \{X \in \mathcal{S}_n \colon x_{ij} \geq 0, \forall i, j\}$,

- the cone of $n \times n$ positive semidefinite matrices: $\mathcal{S}_n^+ = \{X \in \mathcal{S}_n\colon y^T X y \geq 0, \forall y \in \mathbb{R}^n\}$,
- the cone of $n \times n$ copositive matrices: $\mathcal{C}_n = \{X \in \mathcal{S}_n\colon y^T X y \geq 0, \forall y \in \mathbb{R}^n_+\}$,
- the cone of $n \times n$ completely positive matrices:

$$\mathcal{C}_n^* = \{X = \sum_{i=1}^{k} y_i y_i^T,\ k \geq 1,\ y_i \in \mathbb{R}^n_+,\ \forall i = 1, \ldots, k\}.$$

We also use $X \succeq 0$ for $X \in \mathcal{S}_n^+$ and $X \geq 0$ for $X \in \mathcal{N}$. A linear program over $\mathcal{S}_n^+$ is called a semidefinite program while a linear program over $\mathcal{C}_n$ or $\mathcal{C}_n^*$ is called a copositive program.

The sign $\otimes$ stands for Kronecker product. When we consider matrix $X \in \mathbb{R}^{m \times n}$ as a vector from $\mathbb{R}^{mn}$, we write this vector as $\mathrm{vec}(X)$ or $x$. For $u, v \in \mathbb{R}^n$ we define $\langle u, v\rangle = u^T v$ and for $X, Y \in \mathbb{R}^{m\times n}$ we set $\langle X, Y\rangle = \mathrm{trace}(X^T Y)$. For matrix columns and rows we use the matlab notation. Hence $X(i,:)$ and $X(:,i)$ stand for $i$th row and column, respectively, and $X(i:j, p:q)$ is a submatrix of $X$, which consists of elements $x_{st}$, for $i \leq s \leq j$ and $p \leq t \leq q$. If $a \in \mathbb{R}^n$, then $\mathrm{Diag}(a)$ is a $n \times n$ diagonal matrix with $a$ on the main diagonal and $\mathrm{diag}(X)$ is the main diagonal of a square matrix $X$.

For a matrix $Z \in \mathcal{S}_{k^2+1}$, with $k \geq 1$, we often use the following block notation:

$$Z = \left[\begin{array}{c|ccc} Z^{00} & Z^{01} & \cdots & Z^{0k} \\ \hline Z^{10} & Z^{11} & \cdots & Z^{1k} \\ \vdots & \vdots & \ddots & \vdots \\ Z^{k0} & Z^{k1} & \cdots & Z^{kk} \end{array}\right], \tag{1}$$

where $Z^{i0} \in \mathbb{R}^k$, $1 \leq i \leq k$ and $Z^{ij} \in \mathbb{R}^{k\times k}$, $1 \leq i, j \leq k$. Since $Z^{00} \in \mathbb{R}$, we denote it also by $Z_{00}$. Similarly we address components of a matrix $Z \in \mathcal{S}_{k^2}$ via

$$Z = \begin{bmatrix} Z^{11} & \cdots & Z^{1k} \\ \vdots & \ddots & \vdots \\ Z^{k1} & \cdots & Z^{kk} \end{bmatrix}, \tag{2}$$

where $Z^{ij} \in \mathbb{R}^{k\times k}$.

When P is the name of the optimization problem, then $OPT_P$ denotes its optimal value.

## 1.2 Technical preliminaries

In the proof of Theorems 4, 8 and 9, which contain the main results of the paper, we need the following technical lemmas.

**Lemma 1** *Let $Y \in \mathcal{S}_k^+$ with $Y_{ii} = a_i$ and $\sum_{i,j} Y_{ij} = (\sum_i \sqrt{a_i})^2$. Then $Y_{ij} = \sqrt{a_i a_j}$, for $1 \leq i, j \leq k$, or equivalently, $Y = yy^T$ for $y = (\sqrt{a_i})_i$.*

**Proof:** Since $Y \succeq 0$ we know that $|Y_{ij}| \leq \sqrt{Y_{ii}Y_{jj}} = \sqrt{a_i a_j}$ and $\sum_{i,j} Y_{ij} \leq \sum_{i,j} |Y_{ij}| \leq \sum_{i,j} \sqrt{a_i a_j} = (\sum_i \sqrt{a_i})^2$. The equality holds throughout if and only if $Y_{ij} = \sqrt{a_i a_j}$. □

**Lemma 2** *Let*

$$\tilde{Y} = \begin{bmatrix} Y^{11} & Y^{12} \\ Y^{21} & Y^{22} \end{bmatrix} \in \mathcal{S}_{2n}^{+}$$

*with* $Y^{11} = \mathrm{Diag}(a) \in \mathcal{S}_n^+$, $Y^{12} \in \mathbb{R}^{n\times n}$ *and* $Y^{22} = \mathrm{Diag}(b) \in \mathcal{S}_n^+$*. If* $u^T a = \alpha^2$, $u^T b = \beta^2$ *and* $u^T Y^{12} u = \alpha\beta$*, then* $Y^{12}u = \beta/\alpha \cdot a$ *and* $u^T Y^{12} = \alpha/\beta \cdot b$*.*

**Proof:** Without loss of generality we can assume $a_i > 0$ and $b_i > 0$, for all $i$. From $Y \succeq 0$ it follows by using Schur complement [9, Theorem 7.7.6] that $Y^{11} - Y^{12}(Y^{22})^{-1}Y^{21} \succeq 0$, hence $u^T(Y^{11} - Y^{12}(Y^{22})^{-1}Y^{21})u \geq 0$. But

$$\begin{aligned} u^T(Y^{11} - Y^{12}(Y^{22})^{-1}Y^{21})u &= \alpha^2 - \sum_{i=1}^n \frac{(Y^{21}(i,:)u)^2}{b_i} \\ = \alpha^2 - \sum_{i=1}^n \Big(\frac{Y^{21}(i,:)u}{b_i}\Big)^2 b_i &\leq \alpha^2 - \frac{\Big(\sum_{i=1}^n Y^{21}(i,:)u\Big)^2}{\sum_i b_i} \\ = \alpha^2 - \frac{\alpha^2\beta^2}{\beta^2} &= 0 \end{aligned}$$

with equality holding if and only if $Y^{21}(i,:)u/b_i = Y^{21}(j,:)u/b_j,\ \forall i,j$. Since

$$\begin{aligned} \alpha\beta &= \sum_i Y^{21}(i,:)u = \sum_i \frac{Y^{21}(i,:)u}{b_i} b_i \\ &= \frac{Y^{21}(1,:)u}{b_1} \sum_i b_i = \frac{Y^{21}(1,:)u}{b_1}\beta^2 \end{aligned}$$

it follows $Y^{21}(1,:)u/b_1 = \alpha/\beta$ and consequently $Y^{21}u = \alpha/\beta \cdot b$. The second part of the lemma follows by using $Y^{22} - Y^{21}(Y^{11})^{-1}Y^{12} \succeq 0$. □

## 2 Quadratic assignment problem as a copositive program

In this section we first formulate QAP as a quadratically constrained quadratic problem and then dualize it. For the resulting problem, which is a copositive program, we show zero duality gap, which is the main result of the section.

Every permutation matrix has in each row and column exactly one non-zero element, which is equal to 1. Therefore the rows and columns are orthonormal. In fact, this is already a complete characterization of the set of permutation matrices: $\Pi = \{X \in \mathbb{R}^{n\times n} \colon X^TX = XX^T = I,\ X \geq 0\}$ (note that the constraint $XX^T = I$ follows from $X^TX = I$, but we keep it since it becomes important on the dual side. QAP can therefore be rewritten as

$$\min \{\langle X,\, AXB + C\rangle \colon X^TX = XX^T = I,\ X \geq 0\}. \tag{3}$$

In the sequel we use the facts that $\langle C,\, X\rangle = \langle \mathrm{Diag}(c),\, xx^T\rangle$ for $X \in \Pi$ and $\langle X,\, PXQ\rangle = \langle Q^T \otimes$

$P, xx^T\rangle$, for any $X$, and then dualize the orthogonality constraints.

$$\begin{aligned} OPT_{QAP} &= \min_{X\geq 0} \left\{\langle X,\, AXB+C\rangle + \max_{S,T\in\mathcal{S}_n} \{\langle S,\, I-XX^T\rangle + \langle T,\, I-X^TX\rangle\}\right\} \\ &\geq \max_{S,T\in\mathcal{S}_n} \left\{\operatorname{trace}(S)+\operatorname{trace}(T) + \min_{x\in\mathbb{R}^{n^2}_+} \{x^T(B\otimes A+\operatorname{Diag}(c)-I\otimes S-T\otimes I)x\}\right\} \\ &= \max \left\{\operatorname{trace}(S)+\operatorname{trace}(T)\colon S,\ T\in\mathcal{S}_n,\ B\otimes A+\operatorname{Diag}(c)-I\otimes S-T\otimes I\in\mathcal{C}_{n^2}\right\} \\ &= \min \left\{\langle B\otimes A+\operatorname{Diag}(c),\, Y\rangle\colon Y\in\mathcal{C}^*_{n^2},\ \textstyle\sum_i Y^{ii}=I,\ \langle I,\, Y^{ij}\rangle=\delta_{ij},\ \forall i,j\right\}. \end{aligned}$$

We denote the last problem by $QAP_{LD}$. In the last two equations in $QAP_{LD}$ we use the block description of $Y$, introduced in (2).

The first inequality above is due to exchanging min and max. The second equality follows from the fact that the inner minimization problem is bounded from below on the nonnegative orthant if and only if the matrix $B\otimes A+\operatorname{Diag}(c)-I\otimes S-T\otimes I$ is copositive (this is exactly the definition of copositive matrices). The last two problems are dual to each other. The last equality above follows from strict feasibility of the last but one problem, i.e., for $T=S=-\alpha I$ the matrix $B\otimes A+\operatorname{Diag}(c)-I\otimes S-T\otimes I$ is positive definite for $\alpha$ sufficiently large, hence in the interior of $\mathcal{C}_{n^2}$ and therefore strictly feasible, so strong duality holds.

The first inequality may be strict, as follows from the following lemma.

**Lemma 3** *There may exist arbitrary large duality gap between the optimal values $OPT_{QAP}$ and $OPT_{LD}$.*

**Proof:** Let $A=\alpha J$, $B=\beta(J-I)$ and $C=0$, where $\alpha,\,\beta>0$. For every permutation matrix $X$ we have $\langle X,\, AXB+C\rangle=\alpha\beta n(n-1)$, hence $OPT_{QAP}=\alpha\beta n(n-1)$.

The matrix $B\otimes A+\operatorname{Diag}(c)$ and every feasible matrix $Y$ for $QAP_{LD}$ have only nonnegative entries, hence $OPT_{LD}\geq 0$. If we take $Y^*=1/n\, I_{n^2}$, then $Y^*$ is feasible for the $QAP_{LD}$ and $\langle B\otimes A+\operatorname{Diag}(c),\, Y^*\rangle=0$. This means that $Y^*$ is optimal and $OPT_{LD}=0$. The duality gap is $\alpha\beta n(n-1)$ and may be arbitrary large. □

One of the reasons for the non-zero duality gap between $OPT_{QAP}$ and $OPT_{LD}$ is that $QAP_{LD}$ has no constraints on the off-diagonal elements of off-diagonal blocks $Y^{ij}$, i. e. there is no constraint that would force non-zero components in the off-diagonal blocks of $Y$. If $Y=xx^T$ and $x$ corresponds to some permutation matrix $X$, then $Y^{ij}$ is exactly $X(:,i)X(:,j)^T$ and we know that this block must have exactly one component equal to 1 and the others must be 0, hence $\langle J,\, Y^{ij}\rangle=1$ and consequently $\sum_{i,j}\langle J,\, Y^{ij}\rangle=n^2$. We add to (3) the constraint

$$n^2 \;=\; \sum_{i,j}\langle J,\, X(:,i)X(:,j)^T\rangle \;=\; \sum_{i,j}\langle J,\, XE_{ij}X^T\rangle \;=\; \langle X,\, JXJ\rangle \tag{4}$$

and dualize it by repeating the procedure from above. The resulting copositive program, denoted by $QAP_{CP}$, is

$$(QAP_{CP})\qquad \begin{array}{ll} \min & \langle B\otimes A+\operatorname{Diag}(c), Y\rangle \\ \text{s. t.} & Y \;\in\; \mathcal{C}^*_{n^2},\ \ \sum_i Y^{ii} \;=\; I, \\ & \langle I,\, Y^{ij}\rangle \;=\; \delta_{ij},\ \ \forall i,j, \\ & \langle J_{n^2},\, Y\rangle \;=\; n^2\,. \end{array}$$

As above we have $OPT_{QAP} \geq OPT_{CP}$, but we will see below that this holds in fact with equality. First we study the feasible set $\mathcal{F}$ for $QAP_{CP}$:

$$\mathcal{F} := \{Y \in \mathcal{C}^*_{n^2} : Y \text{ feasible for } QAP_{CP}\}.$$

We have the following description of $\mathcal{F}$.

**Theorem 4**

$$\mathcal{F} = \mathrm{CONV}\{xx^T : \; x = \mathrm{vec}(X) \text{ for } X \in \Pi\}.$$

**Proof:** The " $\supseteq$ " part is obvious. Let us consider now the opposite direction. Let $Y$ be arbitrary from $\mathcal{F}$. From the definition of the cone $\mathcal{C}^*_{n^2}$ it follows that there exists $r \geq 1$ and non-zero vectors $y^1, \ldots, y^r \in \mathbb{R}^{n^2}_+$ such that $Y = \sum_{k=1}^r y^k (y^k)^T$. We will find numbers $\lambda_i \in [0,1]$ and vectors $x^i \in \mathbb{R}^{n^2}_+$ such that $y^i(y^i)^T = \lambda_i x^i (x^i)^T$, $1 \leq i \leq r$, $\sum_{i=1}^r \lambda_i = 1$ and each $x_i$ is a vector representation of some permutation matrix $X^i$. This will prove the theorem.

We consider each vector $y^k$ as $\mathrm{vec}(Y^k)$ for some $Y^k \in \mathbb{R}^{n \times n}$, therefore we index components of each $y^k$ by two indices such that $y^k(i,j)$ is $(i,j)$th component of $Y^k$. We will also call with abuse of notation the components $y^k(1,i), \ldots, y^k(n,i)$ by "$i$th column" of $y^k$ and components $y^k(j,1), \ldots, y^k(j,n)$ by "$j$th row" of $y^k$.

From $Y \in \mathcal{C}^*_{n^2}$ it follows $Y \geq 0$ and $Y \succeq 0$. Constraints $\sum_i Y^{ii} = I$ and $\langle I, Y^{ij} \rangle = \delta_{ij}$ therefore imply that $Y^{ii}$ is diagonal for all $i$ and $\mathrm{diag}(Y^{ij}) = 0$ for $i \neq j$. Since $y^k \geq 0$, the same is true for the blocks of $y^k(y^k)^T$. This means in particular that each $y^k$ has in each "row" and "column" one non-zero element at most.

We may write $Y = \sum_{i,j} E_{ij} \otimes Y^{ij}$ and $Y \succeq 0$ implies that the matrix $\tilde{Y}$, defined by

$$\tilde{Y} = (I \otimes u_n^T) Y (I \otimes u_n^T)^T = \sum_{i,j} \langle J, Y^{ij} \rangle E_{ij},$$

is positive semidefinite and satisfies the assumptions of Lemma 1. Therefore we have $\tilde{Y}_{ij} = \langle J, Y^{ij} \rangle = 1$.

Let us fix $i$ and $j$, $i \neq j$, and denote by $a_k$ and $b_k$ the only non-zero components of $y^k$ in $i$th and $j$th column (if e. g. the $i$th column of $y^k$ is 0 then we simply set $a_k = 0$). The $(i,i)$th block of $y^k(y^k)^T$ is therefore diagonal and has one non-zero component at most, which is exactly $a_k^2$ and lies on the main diagonal of the block. Similarly the $(i,j)$th block of $y^k(y^k)^T$ has one non-zero component $a_k b_k$ at most. If it is not 0, then it is off-diagonal in the block. For chosen $i \neq j$ the matrix $Y$ therefore satisfies:

$$\begin{array}{rcccl} 1 & = & \langle I, Y^{ii} \rangle & = & \sum_{k=1}^r a_k^2 \\ 1 & = & \langle I, Y^{jj} \rangle & = & \sum_{k=1}^r b_k^2 \\ 1 & = & \langle J, Y^{ij} \rangle & = & \sum_{k=1}^r a_k b_k \end{array}$$

The Cauchy-Schwarz inequality [9, p. 15], applied to vectors $a = (a_1, \ldots, a_r)$ and $b = (b_1, \ldots, b_r)$, implies $a_k = b_k$ for $1 \leq k \leq r$. Since $i$ and $j$ were arbitrary and none of $y^k$ is a zero vector, we have that each $y^k$ has in each "row" and "column" exactly one non-zero component and all non-zeros are equal (we keep the notation and denote them by $a_k$). Vectors $x^k = y^k / a_k$ therefore correspond to permutation matrices. Let $\lambda_k = a_k^2$, then $Y = \sum_k y^k (y^k)^T = \sum_k \lambda_k x^k (x^k)^T$ and $\sum_k \lambda_k = \sum_k a_k^2 = 1$. $\square$

QAP problem may be written equivalently as

$$OPT_{QAP} \; = \; \min \; \Big\{ \langle B \otimes A + \mathrm{Diag}(c), \, Y \rangle : Y \in \mathrm{CONV}\{xx^T : \; x = \mathrm{vec}(X) \; \mathit{for} \; X \in \Pi\} \Big\}.$$

The following corollary therefore follows immediately.

**Corollary 5** *The optimal value of QAP is equal to the optimal value of $QAP_{CP}$.*

**Remark 1** *This copositive representation again confirms the importance of copositive programming in combinatorial optimization which was revealed by De Klerk and Pasechnik [10] on the stability number problem and illustrated by Povh and Rendl on graph partitioning problem [18]. De Klerk and Pasechnik proved that computing the stability number of a graph is equivalent to solving a copositive program and then presented a hierarchy of linear and positive semidefinite relaxations, which follow from this approach and are strongly connected with the $\vartheta$-function. Povh and Rendl represented the 3-partitioning problem as a copositive program, showed that the simplest (semidefinite) relaxation of the copositive program is exactly the eigenvalue lower bound from [7] and suggested stronger relaxations which are still computable.*

## 3 A hierarchy of SDP relaxations for QAP

In this section we take the formulation $QAP_{CP}$ as a starting point for tractable relaxations. A simple relaxation is obtained by changing $Y \in \mathcal{C}^*_{n^2}$ to the weaker condition $Y \succeq 0$.

$$(QAP_{AW1}) \qquad \begin{array}{lrcl} \min & \langle B \otimes A + \mathrm{Diag}(c), Y \rangle & & \\ \text{s. t.} & Y & \in & \mathcal{S}^+_{n^2}, \\ & \sum_i Y^{ii} & = & I, \\ & \langle I, Y^{ij} \rangle & = & \delta_{ij}, \ \forall i,j, \\ & \langle J_{n^2}, Y \rangle & = & n^2. \end{array}$$

This relaxation corresponds to the Anstreicher-Wolkowicz relaxation for QAP [2], modified by the single equation

$$\langle J_{n^2}, Y \rangle = n^2.$$

We therefore denote it by $QAP_{AW1}$. It is remarkable that this single additional equation often yields a substantial improvement of the bound, see Section 7 below. We note in particular that this SDP has only $O(n^2)$ equality constraints.

A systematic way to replace the intractable constraint $Y \in \mathcal{C}^*_{n^2}$ with weaker tractable constraints was recently suggested by Parrilo. Parrilo [15] (see also [10]) has proposed a hierarchy of cones $\mathcal{K}_0 \subset \mathcal{K}_1 \subset \cdots \subset \mathcal{C}_n$, where

$$\mathcal{K}_r = \{X \in \mathcal{S}_n \colon (\sum_{i,j} x_{ij} z_i^2 z_j^2)(\sum_i z_i^2)^r \text{ is a sum of squares}\},$$

which approximates the copositive cone $\mathcal{C}_n$ from the inside point-wise arbitrarily close. The hierarchy of dual cones approximates the cone $\mathcal{C}^*_n$ from the outside. We will focus on these cones. The first member in this hierarchy is the cone of symmetric doubly nonnegative matrices $\mathcal{K}^*_0 = \mathcal{N}_n \cap \mathcal{S}^+_n$. Our next model will therefore be:

$$(QAP_{\mathcal{K}^*_0}) \qquad \begin{array}{lrcl} \min & \langle B \otimes A + \mathrm{Diag}(c), Y \rangle & & \\ \text{s. t.} & Y & \in & \mathcal{N}_{n^2} \cap \mathcal{S}^+_{n^2}, \\ & \sum_i Y^{ii} & = & I, \\ & \langle I, Y^{ij} \rangle & = & \delta_{ij}, \ \forall i,j, \\ & \langle J_{n^2}, Y \rangle & = & n^2. \end{array}$$

We have to emphasize that this is already an expensive model since the constraint $Y \in \mathcal{N}_{n^2}$ implies $O(n^4)$ linear inequalities.

Trading quality of the relaxation for more computational efficiency, we follow the approach from Zhao et al. [24], and observe the following zero pattern for matrices feasible for $QAP_{\mathcal{K}_0^*}$:

$$Y^{ii}_{jk} = 0, \quad Y^{jk}_{ii} = 0 \quad \forall j \neq k, \forall i.$$

Collecting all these $O(n^3)$ equations symbolically in the map $\mathcal{G}(Y) = 0$, we get the relaxation $QAP_{ZKRW1}$. (We use the acronym ZKRW1 to emphasize that this model is inspired by Zhao et al. [24].) We will show in the following sections that this model is in fact equivalent to the 'gangster-model' from [24].

$$(QAP_{ZKRW1}) \qquad \begin{array}{ll} \min & \langle B \otimes A + \mathrm{Diag}(c), Y\rangle \\ \text{s. t.} & \begin{array}{rclrcl} Y & \in & \mathcal{S}^+_{n^2}, & \mathcal{G}(Y) & = & 0, \\ \sum_i Y^{ii}_{jj} & = & 1, & \langle I, Y^{jj}\rangle & = & 1, \ \forall j, \\ \langle J_{n^2}, Y\rangle & = & n^2. & & & \end{array} \end{array}$$

This relaxation has 'only' $O(n^3)$ constraints, but solving it is still a computational challenge. We address this issue in some more detail in Section 7 below.

# 4 Other SDP relaxations for QAP

In this section we review the SDP relaxations introduced in [24] and further investigated in [20]. The key features of this approach consist in lifting the problem into the space of matrices of order $n^2+1$ and using a parametrization which reflects the assignment constraints. To be specific, the polytope

$$\mathcal{P} := \mathrm{CONV}\Big\{ \begin{bmatrix} 1 \\ x \end{bmatrix} \cdot \begin{bmatrix} 1 \\ x \end{bmatrix}^T : x = \mathrm{vec}(X),\ X \in \Pi \Big\} \tag{5}$$

is replaced by a larger convex set

$$\hat{\mathcal{P}} := \Big\{ Y \in \mathcal{S}_{n^2+1} \colon \exists Z \in \mathcal{S}^+_{(n-1)^2+1} \ \text{ s. t. } Z_{00} = 1 \text{ and } Y = \hat{V} Z \hat{V}^T \Big\} \subset \mathcal{S}^+_{n^2+1},$$

where

$$\hat{V} = \begin{bmatrix} e_0^T \\ W \end{bmatrix},\ W = \Big[\frac{1}{n} u_{n^2} \mid V \otimes V\Big] \text{ and } V = \begin{bmatrix} I_{n-1} \\ \hline -u_{n-1}^T \end{bmatrix}. \tag{6}$$

It is crucial to understand the SDP models which are based on $\hat{\mathcal{P}}$, therefore we add an explicit description of $\hat{\mathcal{P}}$ here. We note however, that the 'only if' part of the result below was already proved in [24].

**Lemma 6** *A matrix $Y \in \mathcal{S}^+_{n^2+1}$ is in $\hat{\mathcal{P}}$ if and only if $Y$ satisfies*

*(i)* $Y_{00} = 1$, $Y^{0i} u = 1$, $1 \le i \le n$, $\sum_{i=1}^n Y_{0i} = u^T$.

*(ii)* $Y^{0j} = u^T Y^{ij}$, $1 \le i, j \le n$.

*(iii)* $\sum_{i=1}^n Y^{ij} = u Y^{0j}$, $1 \le j \le n$.

**Proof:** The "only if " part is done in [24]. We add it here for the sake of completeness. Let $Y = \hat{V}Z\hat{V}^T$. From $Z_{00} = 1$ it follows $Y_{00} = 1$. Let us define operator $\mathcal{T}\colon \mathbb{R}^{(n^2+1)\times(n^2+1)} \to \mathbb{R}^{2n\times(n^2+1)}$ as $\mathcal{T}(X) = \hat{T}X$, where

$$\hat{T} = \left[ \begin{array}{cc} -u_n & I \otimes u_n^T \\ -u_n & u_n^T \otimes I \end{array} \right] \in \mathbb{R}^{2n\times(n^2+1)}.$$

A short exercise shows that $\mathcal{T}(\hat{V}) = 0$, hence $\hat{T}Y = 0$. The second and third property from (i) are just equations $\hat{T}Y(:,0) = 0$ in explicit form. The equations from (ii) are exactly the equations $[-u_n \mid I \otimes u_n^T] \cdot Y(:, 1 : n^2) = 0$ and similarly are equations from (iii) obtained by expanding constraint $[-u_n \mid u_n^T \otimes I] \cdot Y(:, 1 : n^2) = 0$.

Let us consider the opposite direction. Let $Y \in \mathcal{S}^+_{n^2+1}$ satisfy (i)–(iii). Then we have $\hat{T}Y = 0$, hence columns of $Y$ are in $\mathrm{Ker}(\hat{T})$. Since $\mathrm{Ker}(\hat{T})$ is spanned by the columns of $\hat{V}$, which are also linearly independent (see [24, Theorem 3.1]), there exists $\Lambda \in \mathbb{R}^{((n-1)^2+1)\times(n^2+1)}$ such that $Y = \hat{V}\Lambda = \Lambda^T\hat{V}^T$ ($Y$ is also symmetric). In addition, we can find the matrix $\hat{V}^{-1} \in \mathbb{R}^{((n-1)^2+1)\times(n^2+1)}$ such that $\hat{V}^{-1}\hat{V} = I_{(n-1)^2+1}$. Therefore we have $\Lambda = \hat{V}^{-1}\Lambda^T\hat{V}^T$ and $Y = \hat{V}\Lambda = \hat{V}(\hat{V}^{-1}\Lambda^T)\hat{V}^T$, which means that $Y$ is equal to $\hat{V}Z\hat{V}^T$ for $Z = \hat{V}^{-1}\Lambda^T$. From $Y \succeq 0$ and $Y_{00} = 1$ it follows $Z \succeq 0$ and $Z_{00} = 1$. □

**Remark 2** *In Lemma 6 we used $Y \succeq 0$ only to prove $Z \succeq 0$. Hence if $Y \in \mathcal{S}_{n^2+1}$ is not positive semidefinite and satisfies (i)–(iii) from Lemma 6, then we can still write it as $Y = \hat{V}Z\hat{V}^T$ for $Z \in \mathcal{S}_{(n-1)^2+1}$ with $Z_{00} = 1$.*

In the sequel we list some SDP relaxations, summarized from [20, 24]. They are obtained by considering QAP, lifted into the space $\mathcal{S}_{n^2+1}$ as follows:

$$OPT_{QAP} \;=\; \min\,\{\langle L,\, Y\rangle \colon Y \in \mathcal{P}\}, \tag{7}$$

where

$$L = \begin{bmatrix} 0 & \frac{1}{2}c^T \\ \frac{1}{2}c & B \otimes A \end{bmatrix}. \tag{8}$$

To get relaxations, the constraint $Y \in \mathcal{P}$ is replaced by $Y \in \hat{\mathcal{P}}$ and some cutting planes are added.

We follow the notation from [20] and denote them by $QAP_{R_0}$, $QAP_{R_2}$ and $QAP_{R_3}$:

$$\begin{array}{ll} (QAP_{R_0}) & \min\,\{\langle L,\, Y\rangle \colon Y \in \hat{\mathcal{P}},\ \mathrm{Arrow}(Y) = 0\}, \\ (QAP_{R_2}) & \min\,\{\langle L,\, Y\rangle \colon Y \in \hat{\mathcal{P}},\ \mathrm{Arrow}(Y) = 0,\ \mathcal{G}(Y) = 0\}, \\ (QAP_{R_3}) & \min\,\{\langle L,\, Y\rangle \colon Y \in \hat{\mathcal{P}},\ \mathrm{Arrow}(Y) = 0,\ \mathcal{G}(Y) = 0,\ Y \geq 0\}. \end{array}$$

All optimization problems above are semidefinite programs. The constraint $\mathrm{Arrow}(Y) = 0$ demands that in the matrix $Y \in \mathcal{S}_{n^2+1}$ with the block structure from (1) the first row must be equal to the diagonal, i. e. $Y^{0i} = \mathrm{diag}(Y^{ii})$, $1 \leq i \leq n$. The constraint $\mathcal{G}(Y) = 0$ is exactly the same as in $QAP_{ZKRW1}$ and assures the right zero pattern in the right-lower block of $Y$. The new constraint in $QAP_{R_3}$ is due to the observation that any matrix from $\mathcal{P}$ has only non-negative components.

# 5 Lovász-Schrijver relaxation for QAP

In this section we shortly review the Lovász-Schrijver hierarchy of relaxations applied to QAP and recall the SDP model for QAP from Burer and Vandenbussche [3]. Burer and Vandenbussche [3] report good computational results in solving relaxations for QAP, based on the Lovász-Schrijver hierarchy of relaxations for general 0-1 polyhedra [12, 14].

Let $K \subset \mathbb{R}^{n^2}$ be the convex set of doubly stochastic matrices in a vector representation:

$$K = \{x \colon x = \mathrm{vec}(X),\, Xu = u,\, X^T u = u,\, X \geq 0\}.$$

We may express $K$ also as $K = \{x \colon Ax = u_{2n},\ x \geq 0\}$ for

$$A \;=\; \begin{bmatrix} I \otimes u_n^T \\ u_n^T \otimes I \end{bmatrix} \in \mathbb{R}^{2n \times n^2}. \tag{9}$$

The intersection $K \cap \{0,\, 1\}^{n^2}$ is exactly the set of all permutation matrices in a vector form.

Following Lovász and Schrijver we may get a hierarchy of linear and SDP relaxations for the following 0-1 polytope

$$P := \mathrm{CONV}\{K \cap \{0,\, 1\}^{n^2}\} = \mathrm{CONV}\{x \colon x = \mathrm{vec}(X),\, X \in \Pi\}.$$

The first members of these hierarchies are

$$\begin{aligned} N(K) \;&:=\; \left\{ x \in \mathbb{R}^n \colon \begin{bmatrix} 1 \\ x \end{bmatrix} = \mathrm{diag}(Y) \text{ for some } Y \in M(K) \right\} \\ N_+(K) \;&:=\; \left\{ x \in \mathbb{R}^n \colon \begin{bmatrix} 1 \\ x \end{bmatrix} = \mathrm{diag}(Y) \text{ for some } Y \in M_+(K) \right\}, \end{aligned}$$

where

$$\begin{aligned} M(K) \;&:=\; \{Y \in \mathcal{S}_{n^2+1} \colon Ye_0 = \mathrm{diag}(Y),\ Ye_i \in \hat{K},\ Y(e_0 - e_i) \in \hat{K},\ i = 1, \ldots, n^2\} \\ M_+(K) \;&:=\; M(K) \cap \mathcal{S}^+_{n^2+1} \ \text{ and } \ \hat{K} \;:=\; \{\lambda \begin{bmatrix} 1 \\ x \end{bmatrix} : \lambda \geq 0,\, x \in K\}. \end{aligned} \tag{10}$$

We can get higher order linear and SDP relaxations for $K$ recursively:

$$\begin{aligned} N^k(K) \;&:=\; N(N^{k-1}(K)),\ \text{with } N^1(K) = N(K), \\ N_+^k(K) \;&:=\; N_+(N_+^{k-1}(K)),\ \text{with } N_+^1(K) = N_+(K). \end{aligned}$$

In the case of QAP we have a quadratic objective function, hence we are not interested in linear and SDP relaxations for $P$ but we need relaxations for $\mathcal{P}$ from (5). The hierarchy from above yields the following linear relaxation for $\mathcal{P}$

$$\{Y \in \mathcal{S}_{n^2+1} \colon Y \in M(K),\, Y_{00} = 1\}$$

and SDP relaxation for $\mathcal{P}$

$$\{Y \in \mathcal{S}_{n^2+1} \colon Y \in M_+(K),\, Y_{00} = 1\}.$$

The SDP model from [3] (we denote it by $QAP_{L-S}$) is obtained by taking the SDP relaxation from above

$$(QAP_{L-S}) \qquad \min\ \{\langle L,\, Y\rangle \colon Y \in \mathcal{S}^+_{n^2+1},\ Ye_i \in \hat{K},\ Y(e_0 - e_i) \in \hat{K},\ i = 1, \ldots, n^2,\ Y_{00} = 1\},$$

where $L$ is from (8).

**Remark 3** *For our particular $K$ we may replace in $QAP_{L-S}$ constraints $Ye_i \in \hat{K},\ Y(e_0 - e_i) \in \hat{K}$, $i = 1, \ldots, n^2$, by the following linearly independent set of constraints $Ye_i \in \hat{K},\ i = 0, 1, \ldots, n^2$.*

## 6 Comparing the relaxations

In this section we show that $QAP_{ZKRW1}$ and $QAP_{R_2}$ are equivalent and $QAP_{\mathcal{K}_0^*}$, $QAP_{R_3}$ and $QAP_{L-S}$ are also equivalent. The difference in favor of $OPT_{L-S}$, noticed in [3], is therefore due to computational reasons (Sotirov [20] used the bundle method which is known to be very slow close to the optimum, therefore Sotirov probably stopped to soon).

We need the following lemma.

**Lemma 7** *A matrix $Y \in \mathcal{S}^+_{n^2}$ is feasible for $QAP_{ZKRW1}$ if and only if $Y$ satisfies*

*(i)* $\mathcal{G}(Y) = 0$, $\operatorname{trace}(Y^{ii}) = 1$ *for* $1 \le i \le n$, $\sum_i \operatorname{diag}(Y^{ii}) = u$,

*(ii)* $u^T Y^{ij} = \operatorname{diag}(Y^{jj})^T$ *for* $1 \le i, j \le n$,

*(iii)* $\sum_i Y^{ij} = u \operatorname{diag}(Y^{jj})^T$ *for* $1 \le j \le n$.

**Proof:** If $Y$ satisfies (i)–(iii), then obviously $Y$ is feasible for $QAP_{ZKRW1}$ (feasibility for all but last constraint follows from (i), while the last is a simple corollary of (i) and (ii)).

The opposite direction is less obvious. Let $Y \in \mathcal{S}^+_{n^2}$ be feasible for $QAP_{ZKRW1}$. Property (i) contains only constraints from $QAP_{ZKRW1}$, hence is satisfied.

The property (ii) follows from the fact that the matrix

$$\tilde{Y} = \sum_{i,j} \langle J, Y^{ij} \rangle \cdot E_{ij} = (I \otimes u^T)\, Y\, (I \otimes u^T)^T$$

is positive semidefinite and satisfies all assumptions from Lemma 1, therefore we have $\tilde{Y}_{ij} = 1$ (we used this fact also in the proof of Theorem 4). This implies that for any $i \neq j$ the matrix

$$\begin{bmatrix} Y^{ii} & Y^{ij} \\ Y^{ji} & Y^{jj} \end{bmatrix}$$

satisfies presumptions of Lemma 2, hence the property (ii) follows.

We prove (iii) by considering $\hat{Y} = \sum_{i,j} Y^{ij} = (u^T \otimes I)\, Y\, (u^T \otimes I)^T$ and repeating the arguments from the previous paragraph. □

We have now the tools to compare the semidefinite and Lovász–Schrijver relaxations for QAP.

**Theorem 8** *The SDP model $QAP_{R_2}$ is equivalent to the model $QAP_{ZKRW1}$ in the sense that feasible sets are in bijective correspondence and $OPT_{R_2} = OPT_{ZKRW1}$.*

**Proof:** First we show that for any feasible $Y \in \mathcal{S}^+_{n^2+1}$ for $QAP_{R_2}$ we can find exactly one matrix $Z = Z(Y) \in \mathcal{S}^+_{n^2}$, feasible for $QAP_{ZKRW1}$ and vice versa. We address the components of $Y$ and $Z$ via the block structure, described in (1) and (2). The correspondence is as follows:

$$Y \mapsto Z = Z(Y) = [Y^{ij}]_{1 \le i,j \le n} \;\text{ and }\; Z \mapsto Y = Y(Z) = \begin{bmatrix} 1 & z^T \\ z & Z \end{bmatrix}, \quad z = \operatorname{diag}(Z). \tag{11}$$

If $Y$ is feasible for $QAP_{R_2}$, then $Z = Z(Y) \succeq 0$ and Lemma 6 implies that $Z$ satisfies (i)–(iii) from Lemma 7, hence $Z$ is feasible for $QAP_{ZKRW1}$.

Let $Z$ be feasible for $QAP_{ZKRW1}$. Then we have $\operatorname{Arrow}(Y) = 0$ and from Lemma 7 it follows that $Y$ satisfies (i)–(iii) from Lemma 6, hence by using Remark 2 we have $Y = \hat{V} R \hat{V}^T$ for some

$R \in \mathcal{S}_{(n-1)^2+1}$ with $R_{00} = 1$. It remains to show that $Y \succeq 0$. From the block structure of $Y$ and $\hat{V}$ it follows that

$$Y = \begin{bmatrix} 1 & z^T \\ z & Z \end{bmatrix} = \begin{bmatrix} e_0^T R e_0 & e_0^T R W^T \\ W R e_0 & W R W^T \end{bmatrix},$$

where $W$ is from (6). Since $Z = WRW^T \succeq 0$, we have $R \succeq 0$ and consequently $Y \succeq 0$.

For any pair $(Y, Z)$ of feasible solutions for $QAP_{R_2}$ and $QAP_{ZKRW1}$, which satisfy (11), we have

$$\langle B \otimes A + \mathrm{Diag}(c), Z \rangle = \langle L, Y \rangle.$$

The equality $OPT_{R_2} = OPT_{ZKRW1}$ is therefore an immediate consequence of the first part of the proof. □

In the following theorem we compare SDP models $QAP_{R3}$, $QAP_{\mathcal{K}_0^*}$ and $QAP_{L-S}$ .

**Theorem 9** *The SDP models $QAP_{\mathcal{K}_0^*}$, $QAP_{R_3}$ and $QAP_{L-S}$ are equivalent, i. e. the feasible sets are in bijective correspondence and $OPT_{R_3} = OPT_{\mathcal{K}_0^*} = OPT_{L-S}$.*

**Proof:** The models $QAP_{R_3}$ and $QAP_{\mathcal{K}_0^*}$ are obtained from the models $QAP_{R_2}$ and $QAP_{ZKRW1}$ by adding the sign constraint $Y \geq 0$. The equivalence therefore follows from the equivalence of models $QAP_{R_2}$ and $QAP_{ZKRW1}$, proven in Theorem 8.

It remains to show equivalence of models $QAP_{\mathcal{K}_0^*}$ and $QAP_{L-S}$. Let $Y \in \mathcal{S}^+_{n^2+1}$ be feasible for $QAP_{L-S}$ and $Z$ the matrix, obtained from $Y$ by deleting the first row and column. According to Remark 3 we know that $Y \geq 0$, $Ye_0 = \mathrm{diag}(Y)$ and $Ye_i \in \hat{K}$ for $0 \leq i \leq n^2$ and $\hat{K}$ from (10). Therefore $Z \geq 0$ and if we prove that $Z$ is feasible for $QAP_{ZKRW1}$, then $Z$ is feasible also for $QAP_{\mathcal{K}_0^*}$, since $QAP_{ZKRW1}$ was obtained from $QAP_{\mathcal{K}_0^*}$ by omitting the sign constraint for non-diagonal entries in the non-diagonal blocks.

It is sufficient to show that $Z$ satisfy properties (i)–(iii) from Lemma 7. Constraints $Ye_0 \in \hat{K}$ together with $Y_{00} = 1$ implies that $AY(1 : n^2, 0) = u_{2n}$, where $A$ is from (9). This equations can be written equivalently as $u^T Y^{i,0} = 1$ for $1 \leq i \leq n$ and $\sum_i Y^{i0} = u$. By using $Ye_0 = \mathrm{diag}(Y)$ we get that the main diagonal of $Z$ satisfies the property (i) from Lemma 7. Similarly we see that for any $1 \leq i \leq n^2$ the matrix $Y$ satisfy $Ye_i \in \hat{K}$, or equivalently $AY(1 : n^2, i) = Y(0, i)u_{2n}$. Expanding this terms yields exactly the properties (ii) and (iii) from Lemma 7 for the matrix $Z$. For feasibility for $QAP_{ZKRW1}$ it remains to show $\mathcal{G}(Z) = 0$. Let us consider the diagonal block $Z^{ii}$. Since we know that components of $Z$ are non-negative, the property $u^T Z^{ii} = \mathrm{diag}(Z^{ii})$ implies that $Z^{ii}_{jk} = 0$ for $j \neq k$. Similarly we get from $\sum_i Z^{ij} = u\,\mathrm{diag}(Z^{jj})^T$ that $\sum_i \mathrm{diag}(Z^{ij}) = \mathrm{diag}(Z^{jj})$ which means that all diagonal elements in any non-diagonal block must be zero.

Therefore $Z$ is feasible for $QAP_{\mathcal{K}_0^*}$ and gives the same value of objective function. The reverse direction, i. e. proving that the feasible solution $Z$ for $QAP_{\mathcal{K}_0^*}$ yields a feasible solution $Y$ for $QAP_{L-S}$ via

$$Y = \begin{bmatrix} 1 & z^T \\ z & Z \end{bmatrix}, \quad z = \mathrm{diag}(Z),$$

is easy and again relies on Lemma 7. □

## 7 Summary and practical implications

We now summarize the equivalences shown in the previous sections and draw some practical conclusions. In Table 1, we collect in the same line problems which we showed to be equivalent. The

| problems | | | hardness | equivalence |
|---|---|---|---|---|
| $OPT_{QAP}$ | $OPT_{CP}$ | | NP-hard | Theorem 4 |
| $OPT_{R_3}$ | $OPT_{\mathcal{K}_0^*}$ | $OPT_{L-S}$ | $O(n^4)$ | Theorem 8 |
| $OPT_{R_2}$ | $OPT_{ZKRW1}$ | | $O(n^3)$ | Theorem 9 |
| | $OPT_{AW1}$ | | $n^2+n$ | |
| $OPT_{R_0}$ | | | $n^2+1$ | |

Table 1: Problems in the same line are equivalent

| name | $n$ | $QAP_{R_0}$ | $QAP_{AW1}$ | $QAP_{R_2}$ | $OPT_{QAP}$ |
|---|---|---|---|---|---|
| nug15 | 15 | -823 | 981 | 1069 | 1150 |
| nug20 | 20 | -2073 | 2245 | 2386 | 2570 |
| nug25 | 25 | -4683 | 3234 | 3454 | 3744 |
| nug30 | 30 | -10965 | 5305 | 5695 | 6124 |

Table 2: The new bound $QAP_{AW1}$ has the same time complexity as $QAP_{R_0}$ with much better quality.

last column refers to the theorem which shows the equivalence. All relaxations are formulated in the space of symmetric matrices of order $n^2$ (or $n^2+1$), hence each relaxation has $O(n^4)$ variables. The weakest relaxation has $O(n^2)$ constraints, while the strongest ones all have $O(n^4)$ constraints.

The two weakest, but computationally cheapest models can be solved easily by interior point methods. The other models (with at least $O(n^3)$ constraints) can not be solved by interior point methods, see [20], where the bundle method is suggested to solve both $QAP_{R_2}$ and $QAP_{R_3}$ with low accuracy by considering the Lagrangian dual, which is obtained by dualizing all constraints except those from $QAP_{R_0}$. Thus a function evaluation of the Lagrangian amounts to solving $QAP_{R_0}$. In [20] it is reported that after about 150 bundle iterations, i.e. function evaluations of $QAP_{R_0}$, one has a rough estimate of the respective relaxations.

The strongest models are still considered a computational challenge. Currently only the augmented Lagrangian method proposed by Burer and Vandenbussche leads to moderately accurate solutions of $QAP_{L-S}$. The bundle method [20] seems to be slightly faster, but gives less accurate results.

The newly introduced relaxation $QAP_{AW1}$ may lead the way to new computational approaches to get faster and more reliable methods for $QAP_{\mathcal{K}_0^*}$. We provide some preliminary computational comparisons of this bound with some other bounds in Table 2. We use the standard test instances from the Nugent collection, and indicate the size of each instance in column 2. The bounds $QAP_{R_0}$ and $QAP_{AW1}$ come from SDP which have roughly the same number of constraints (see table 1) and can be solved quite efficiently (in a few minutes for $n = 30$) by standard interior point methods. The difference in quality between these two models is quite surprising. We observe in particular that the difference to $QAP_{R_2}$ is not terribly big, while the computational effort for $QAP_{R_2}$ is substantial (several hours using the bundle method from [20] for $n = 30$).

Using the bundle method based on this new model should therefore provide a significant speed-up in the bound computation. This is currently under investigation and computational results will be reported elsewhere.

## Acknowledgement

The authors are grateful to Renata Sotirov for making her MATLAB code available to us.

## References

[1] K. Anstreicher: Recent Advances in the Solution of Quadratic Assignment Problems. Math. Program. B, 97:27–42, 2003.

[2] K. Anstreicher, H. Wolkowicz: On Lagrangian Relaxation of Quadratic Matrix Constraints. SIAM J. Matrix Anal. Appl., 22:41–55, 2000.

[3] S. Burer and D. Vandenbussche: Solving lift-and-project relaxations of binary integer programs. SIAM J. Optim. 16:726-750, 2006.

[4] R. E. Burkard, F. Çela, S. E. Karisch, F. Rendl: QAPLIB - A Quadratic Assignment Problem Library. Available at `http://www.seas.upenn.edu/qaplib/`, February 2006.

[5] F. Çela: *The Quadratic Assignment Problem: Theory and Algorithms.* Kluwer Academic Publishers, Dordrecht, 1998.

[6] W. E. Donath, A. J. Hoffman: Lower bounds for the partitioning of graphs, IBM J. Res. Develop. 17:420–425, 1973.

[7] C. Helmberg, F. Rendl, B. Mohar, S. Poljak: A spectral approach to bandwidth and separator problems in graphs. Linear and Multilinear Algebra, 39:73–90, 1995.

[8] A. J. Hoffman, H. W. Wielandt: The variation of the spectrum of a normal matrix. Duke Math. J. 20:37–39, 1953.

[9] R. A. Horn, C. R. Johnson: *Matrix Analysis.* Cambridge University Press, 1985.

[10] E. De Klerk, D. V. Pasechnik: Approximation of the stability number of a graph via copositive programming. SIAM J. Optim., 12:875–892, 2002.

[11] T. C. Koopmans, M. J. Beckmann: Assignment problems and the location of economics activities. Econometrica, 25:53–76, 1957.

[12] M. Laurent: A comparison of the Sherali-Adams, Lovász-Schrijver and Lasserre relaxations for 0-1 programming. Math. Oper. Res. 28:470–496, 2003.

[13] E. M. Loiola, N. M. M. de Abreu, P. O. Boaventura-Netto, P. Hahn, T. Querido: A survey for the quadratic assignment problem. Invited Review, European Journal of Operational Research, 176: 657–690.

[14] L. Lovász, A. Schrijver: Cones of matrices and set-functions and 0-1 optimization. SIAM J. Optim. 1:166–190, 1991.

[15] P. A. Parrilo: Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization. PhD Thesis, California Institute of Technology, Pasadena, 2000.

[16] C. H. Papadimitriou, K. Steiglitz: *Combinatorial Optimization: Algorithms and Complexity.* Dover, 1998.

[17] J. Povh: Application of semidefinite and copositive programming in combinatorial optimization. PhD Thesis, University of Ljubljana, Faculty of mathematics and physics, Ljubljana, 2006.

[18] J. Povh, F. Rendl: A copositive programming approach to graph partitioning, submitted 2005.

[19] F. Rendl: *The quadratic assignment problem.* Appeared as a chapter in the book "Facility Location: Applications and Theory" (Springer 2002), edited by Z. Drezner and H. Hamacher.

[20] F. Rendl, R. Sotirov: Bounds for the quadratic assignment problem using the bundle method. To appear in Math. Program. B.

[21] S. Sahni, T. Gonzales: P–complete approximation problems. J. Assoc. Comput. Mach., 23:555–565, 1976.

[22] R. Sotirov: Bundle methods in Combinatorial Optimization, PhD Thesis, Universitaet Klagenfurt, Klagenfurt, 2003.

[23] M. J. Todd, K. C. Toh, R. H. Tütüncü: SDPT3 version 3.02. Available from `http://www.math.nus.edu.sg/~mattohkc/sdpt3.html`, February 2006.

[24] Zhao, Q., Karisch, S. E., Rendl, F., and Wolkowicz, H.: Semidefinite Programming Relaxations for the Quadratic Assignment Problem. J. Comb. Optim., 2:71-109, 1998.