CO 602: Fundamentals of Optimization
Fall 2010
**Problem Set 5**
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Handed out: 2010-Oct-25.

Due: 2010-Nov 1, in lecture.

1. Apply the instructor's routine `SEFlinprog` to solve the problem called `degenerate.mat` from the course home page. The program will fail badly on this example. (You'll also need SEFlinprog1 and the routine extendtobasis.) Pinpoint the step in the program's execution where something first goes wrong (i.e., the first occasion when a mathematical property of the simplex method is violated). Try to explain in as much detail as you can what goes wrong and why.

   Suggestions: You can cut and paste SEFlinprog and extendtobasis from the PDF posted on UWACE (Problem Set 4 Solutions). The other routine SEFlinprog1 is available from the course home page. You can then modify the latter with print statements (i.e., statements consisting of variable names or expressions with no trailing semicolon) to see the values of the variables at each step. One component in the failure of this program is roundoff error, which is the error in the 16th significant decimal place that occurs with IEEE double precision arithmetic. In IEEE double precision, dividing 0/0 yields a quantity denoted as NaN (for "not a number"). Another component in the failure is degeneracy. Unexpected linear dependence also plays a role.

2. Consider the capacitated min-cost flow problem in which the network has just two nodes, $s, t$, and $m$ parallel arcs connecting $s$ to $t$ each with its own cost and capacity. Let the costs be $c_1, \ldots, c_m$ and capacities $u_1, \ldots, u_m$. To simplify notation, assume $c_1 \leq c_2 \leq \cdots \leq c_m$. Let $z$ stand for $b_s$, which equals $-b_t$.

   (a) Suppose you are given a particular value of $z$, $0 \leq z \leq u_1 + \cdots + u_m$. Figure out the optimal solution to this problem, and verify optimality by checking the dual. [Hint: saturate the lowest-cost arcs first.]

   (b) Let $g(z)$ be the optimal value of this objective function, written as a function of $z$. (Naturally, $g(z)$ also depends on $c_1, \ldots, c_m$ and $u_1, \ldots, u_m$.) Show that $g$ is a piecewise linear continuous convex function over the interval $[0, u_1 + \cdots + u_m]$.

   (c) Conversely, suppose you are given a specified piecewise linear continuous nondecreasing convex function $\phi(x)$ over an interval $[0, u]$, such that $\phi(0) = 0$. The specification takes the form of a list of breakpoints of $\phi$, say $(x_i, y_i)$ for $i = 0, \ldots, m$, where $0 = x_0 < x_1 < \cdots < x_m = u$. Design a network of parallel arcs so that the optimal flow of size $z$ through the network is exactly $\phi(z)$.

3. Consider Phase 1 of the network simplex algorithm applied to an uncapacitated min-cost flow problem. Show how to carry out the steps of Phase 1 using a purely combinatorial algorithm. Those steps are (a) pivoting operations for the Phase 1 artificial

problem, and (b) extension of the $x$-columns to basis in the case that some of the $y$-variables are still in the optimal BFS.

It is suggested that you construct the coefficient matrix for Phase 1 in such a way that it is a truncated node-arc incidence matrix, in which case you can use all the combinatorial algorithms presented in lecture, and you don't have to develop any new combinatorial algorithms. It is suggested that you not change the signs of rows of $A$ as was done in lecture for Phase 1, because this will spoil the property that $A$ is a node-arc incidence matrix. Think of an alternative way to deal with the signs of the $\mathbf{y}$ variables.

Make the same assumptions as in lecture, namely, that the sum of the source/sinks is 0 and that that network is connected in the sense that the underlying undirected graph is connected.

4. Given an uncapacitated min-cost flow problem, assume, as in lecture, that the undirected graph arising from the network is connected and that the sum of the source/sink values is 0.

   Suppose the problem is infeasible. One proof of infeasibility is a vector $\mathbf{y}$ such that $A^T\mathbf{y} \le \mathbf{0}$ and $\mathbf{b}^T\mathbf{y} > 0$ by Farkas' lemma.

   Another certificate of infeasibility is as follows. If there is a nonempty subset $S$ of nodes such that there are no outgoing arcs from $S$ to $N - S$ and such that the sum of $b$-values in $S$ is positive, then the problem is infeasible.

   (a) Show that given a subset $S$ as described in the previous paragraph, one can construct a vector $\mathbf{y}$ that satisfies the conditions $A^T\mathbf{y} \le \mathbf{b}$, and $\mathbf{b}^T\mathbf{y} > 0$.

   (b) Show the other direction: given a $\mathbf{y}$ satisfying $A^T\mathbf{y} \le \mathbf{b}$, and $\mathbf{b}^T\mathbf{y} > 0$, construct a set $S$ of with no outgoing arcs such the sum of $b$-values in $S$ is positive. To simplify notation, assume the vertices are renumbered so that $y_1 \ge y_2 \ge \cdots \ge y_n$. Furthermore, since the columns of $A^T$ are not independent, we can add or subtract a multiple of the vector of all 1's, so assume $y_n = 0$. [Hint: try to find an $S$ of the form $\{1, \ldots, k\}$ for some $k \le n$.]