CO 602: Fundamentals of Optimization
Fall 2010
Problem Set 4
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Handed out: 2010-Oct-13.

Due: 2010-Oct-22 in lecture.

1. Download the m-files `seflinprog1` and `extendtobasis` from the course web page. The first is a Matlab program for solving a linear programming problem in standard equality form given an initial BFS. The second is a routine that takes indices of linearly independent columns of a matrix and extends them to a basis. (It is similar to the program you wrote for PS3, Question 4.) Write a new m-file called `seflinprog` that solves an SEF linear programming instance that uses `seflinprog1` and `extendtobasis` as subroutines and does not require an initial BFS. In other words, `seflinprog1` should take as input arguments $A, \mathbf{b}, \mathbf{c}$ and should return $\mathbf{x}$ and a variable `flag`. The matrix A may be assumed to have linearly independent rows. Here, `flag` should be set to 0 to indicate successful completion of simplex, 1 to indicated unboundedness, and 2 to indicate infeasibility. Thus, part of this question requires you to write Phase 1. Test your algorithm on the test cases on the course website. Hand in a listing of your code and sample runs.

   Note: An issue briefly mentioned in lecture and also in the text on pp. 112–113 is the following. At the end of Phase 1, if the optimal solution has zero objective, then a feasible point has been found. However, the basis that goes with this feasible solution may use some columns from the I submatrix of the Phase 1 constraints. Let B_0 be the columns from A of the final BFS of Phase 1. Thus, the issue is that B_0 may contain fewer than m columns. To obtain a BFS that uses only columns of A, one discards the basis columns that come from I and then extends B_0 to a basis of $\mathbf{R}^m$ using `extendtobasis`.

2. Given an instance of s.e.f. linear programming, say (P), form its dual (D). Now convert (D) to s.e.f. using the techniques described at the beginning of the course (duplication of variables with no sign constraint and introduction of slacks). Call the resulting s.e.f. problem (D'). Finally, form the dual of (D') using the standard formula for dual of an s.e.f. problem; call it (P'). Identify the relationship between (P') and (P).

3. Prove the complementary slackness theorem for s.e.f. linear programming (i.e., that a pair of feasible primal-dual solutions $(\mathbf{x}, \mathbf{y})$ are both optimal if and only if $x(i)(A(:,i)^T\mathbf{y} - c(i)) = 0$ for all i) holds even without using the assumption that $\text{rank}(A) = m$.

   [Hint: Let the problem be denoted (P). By assumption of feasibility, $A\mathbf{x} = \mathbf{b}$ is consistent, and therefore there is an equivalent problem (P') in which A has fewer (but independent) rows. The proof in lecture of complementarity can be applied to (P'), whose dual is (D'). Then show how to deduce an optimizer of (D), the dual of

(P), from the optimizer of (D'), and show that the complementary slackness result for (P', D') applies to (P, D).]

4. Let G be an undirected graph. Assign orientations to its edges arbitrarily, and let A be the resulting node-arc incidence matrix. Let B be a subset of edges of G, which corresponds to a subset of columns of A. Show that $A(:, B)$ has linearly independent columns if and only if B defines an acyclic subgraph of G.