CO 663 – Assignment 1

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Solution to Problem 1.1. This is an alternate solution to problem 1.1, which uses the supporting hyperplane theorem. Also, recall that the level set $S_\alpha(f) := \{x \in \mathbb{E} : f(x) \leq \alpha\}$.
Now, suppose the contrary, and let $x \in \Omega$ such that f is discontinuous at x. I.e., suppose that $\exists \{y^n\}_{n\in\mathbb{N}} \subseteq \Omega$ such that $\lim_{n\to\infty} y^n = x$ but $\lim_{n\to\infty} f(y^n) \neq f(x)$.

Claim 1. $\forall \epsilon > 0$, $x \in \text{int}(S_{f(x)+\epsilon}(f))$.

Proof of Claim 1. Suppose the contrary, i.e. $\exists \epsilon > 0$ such that $x \notin \text{int}(S_{f(x)+\epsilon}(f))$. Fix such an ϵ. Note that $S_{f(x)+\epsilon}(f)$ is convex (because f is convex) and $x \in S_{f(x)+\epsilon}(f) \setminus \text{int}(S_{f(x)+\epsilon}(f))$. Therefore, by the supporting hyperplane theorem, there exists a hyperplane $P = \{y \in \mathbb{E} : \langle y, a\rangle = b\}$ defining a closed half space $X = \{y \in \mathbb{E} : \langle y, a\rangle \leq b\}$ such that $x \in P$ and $S_{f(x)+\epsilon}(f) \subseteq X$.
Since Ω is open, $x + ta \in \Omega$ for sufficiently small $t > 0$. Therefore, there exists $\bar{t} > 0$ such that $x+\bar{t}a \in \Omega$ and $z = x+\bar{t}a \in X^c$ (complement of X). Since $z \notin S_{f(x)+\epsilon}$, we have $f(z) > f(x)+\epsilon$. Therefore, $x, z \in S_{f(z)}$ and hence, by convexity of $S_{f(z)}$, $\{\lambda x+(1-\lambda)z : \lambda \in [0,1]\} \subseteq S_{f(z)} \subseteq \Omega$. For all $\lambda > 0$, $(1-\lambda)x + \lambda z \notin S_{f(x)+\epsilon}$, so $f((1-\lambda)x + \lambda z) > f(x) + \epsilon$. For sufficiently small $\lambda > 0$,

$$(1-\lambda)f(x) + \lambda f(z) < f(x) + \epsilon < f((1-\lambda)x + \lambda z)$$

contradicting convexity of f. □ (Claim 1)

Claim 2. $\forall \epsilon > 0$, $\exists N_\epsilon \in \mathbb{N}$ such that for all $n > N_\epsilon$, $f(y^n) < f(x) + \epsilon$

Proof of Claim 2. By claim 1, $\forall \epsilon > 0$, $x \in \text{int}(S_{f(x)+\epsilon}(f))$. I.e. $\exists \delta_\epsilon > 0$ such that $\forall z \in x + \delta_\epsilon B$, $f(z) < f(x) + \epsilon$. $\lim_{n\to\infty} y^n = x$, so $\exists N_\epsilon \in \mathbb{N}$ such that for all $n > N_\epsilon$, $\|y^n - x\| < \delta_\epsilon \Rightarrow y^n \in x + \delta_\epsilon B \Rightarrow f(y^n) < f(x) + \epsilon$. □ (Claim 2)

Claim 3. $\exists \mu > 0$, $\forall N \in \mathbb{N}$, $\exists n_{\mu,N} > N$ such that $f(y^{n_{\mu,N}}) < f(x) - \mu$.

Proof of Claim 3. Suppose the contrary. I.e., $\forall \mu > 0$, $\exists N_\mu \in \mathbb{N}$ such that $\forall n > N_\mu$, $f(y^n) \geq f(x)-\mu$. By claim 2, $\exists N'_\mu \in \mathbb{N}$ such that $\forall n > N'_\mu$, $f(y^n) < f(x)+\mu$. Let $N''_\mu = \max\{N_\mu, N'_\mu\}$. Then for all $n > N''_\mu$, $\|f(y^n) - f(x)\| < \mu$. This being true for all $\mu > 0$, $\lim_{n\to\infty} f(y^n) = f(x)$, contradiction. □ (Claim 3)

Claim 4. $\exists \mu > 0$, $\exists \delta > 0$, $\exists n \in \mathbb{N}$ such that

(i) $f(y^n) < f(x) - \mu$
(ii) $y^n \in x + \delta B$
(iii) $\forall z \in x + \delta B$, $f(z) < f(x) + \mu$

Proof of Claim 4. By claim 3, $\exists \mu > 0$, $\forall N \in \mathbb{N}$, $\exists n > N$ such that $f(y^n) < f(x) - \mu$. Fix such a μ. By claim 1, $x \in \operatorname{int} S_{f(x)+\mu}$, i.e. $\exists \delta > 0$ such that $\forall z \in x + \delta B$, $f(z) < f(x) + \mu$. Fix such a δ, and note that the choice of μ and δ satisfy (iii). Because $\lim_{n\to\infty} y^n = x$, $\exists N \in \mathbb{N}$ such that $\forall n > N$, $y^n \in x + \delta B$. Therefore, by claim 3, $\exists n > N$ such that $f(y^n) < f(x) - \mu$ and $y^n \in x + \delta B$, so μ, δ and n satisfy (i) and (ii). □ (Claim 4)

Fix μ, δ and n with the properties (i)-(iii) in claim 4. Let

$$z = 2x - y^n = x - (y^n - x)$$

Note that $\|y^n - x\| \le \delta$ (by claim 4, property (ii)), so $z \in \delta B + x$. Hence (by claim 4, property (iii)), $f(z) < f(x) + \mu$. Also (by claim 4, property (i)), $f(y^n) < f(x) - \mu$, so

$$\frac{1}{2} f(y^n) + \frac{1}{2} f(z) < \frac{1}{2}(f(x) - \mu) + \frac{1}{2}(f(x) + \mu) = f(x)$$

But $x = \frac{1}{2} y^n + \frac{1}{2} z$, so this contradicts convexity of f. □ (Problem 1.1)

Theorem (Supporting Hyperplane Theorem). If $C \subseteq \mathbb{R}^n$ and $x \in C \setminus \operatorname{int}(C)$, then there exists a hyperplane $P = \{y \in \mathbb{R}^n : \langle y, a \rangle = b\}$ such that $\langle x, a \rangle = b$ and $\langle y, a \rangle \le b$ for all $y \in C$ (i.e. C lies in a closed half space defined by P).

Proof of Theorem. In finite dimensions, at least, this theorem follows easily from the hyperplane separation theorem. $x \notin \operatorname{int}(C)$, so there is a sequence $\{z^n \notin C\}_{n \in \mathbb{N}}$ which converges to x. For each n, find a hyperplane $P_n = \{y \in \mathbb{R}^n : \langle y, a_n \rangle = b_n\}$ separating z^n from C (hyperplane separation theorem). Assume wlog that $\langle y, a_n \rangle \le b_n$ for all $y \in C$. Also assume wlog that $\|a_n\| = 1$. The unit sphere is a compact manifold, so a_n has a convergent subsequence. Thus, (by throwing out everything not in a particular convergent subsequence) assume wlog that $a = \lim_{n\to\infty} a_n$ exists. Note that P_n separates x from z_n, so $\lim_{n\to\infty} d(x, P_n) = 0$ (where $d(x, P_n) = \min\{\|x - y\| : y \in P_n\}$). Hence

$$b = \lim_{n\to\infty} b_n = \lim_{n\to\infty} \langle x, a_n \rangle = \langle x, a \rangle$$

Finally, for all $y \in C$,

$$\langle y, a \rangle = \lim_{n\to\infty} \langle y, a_n \rangle \le \lim_{n\to\infty} b_n = b \quad \square \text{ (Theorem)}$$