The dual simplex algorithm

STEP 1. To employ this algorithm, the problem must be dual feasible, that is, all $z_j - c_j \geq 0$. If this condition is met, go to Step 2. (The following section will discuss one method of attaining a dual feasible solution.)

STEP 2. *Determine the departing variable.* If $\beta_i \geq 0$, for all i, then the current solution is optimal; stop. Otherwise, select the row associated with the most negative β_i. Denote this row as row r. The basic variable $x_{B,r}$ associated with this row is the departing variable.

STEP 3. *Check for primal feasibility.* If $\alpha_{r,j} \geq 0$, for all j, then the primal problem is infeasible and the dual problem has an unbounded objective; stop. Otherwise, go to Step 4.

STEP 4. *Determine the entering variable.* Use the following minimum ratio test to determine the entering basic variable. That is, let

$$\frac{z_k - c_k}{-\alpha_{r,k}} = \text{minimum}\left\{\frac{z_j - c_j}{-\alpha_{r,j}} : \alpha_{r,j} < 0\right\}$$

Column k is the pivot column, $\alpha_{r,k}$ is the pivot element, and the basic variable x_k associated with column k is the entering variable. Go to Step 5.

STEP 5. *Pivot and establish a new tableau*

(a) The entering variable x_k is the new basic variable in row r.

(b) Use elementary row operations on the old tableau so that the column associated with x_k in the new tableau consists of all zero elements except for a 1 at the pivot position $\alpha_{r,k}$.

(c) Return to Step 2.

The dual simplex algorithm is generally considered unequal to the task of performing as a general-purpose linear programming algorithm because of the difficulty in finding an initial basic solution that is dual feasible. Consequently, the following example obviously has been contrived so as to exploit the dual simplex properties. The reader should recognize the improbability of finding such problems in practice.

Example 6.10: The Dual Simplex Method

$$\text{minimize } z = 8x_1 + 5x_2 \tag{6.236}$$

subject to

$$x_1 + x_2 \geq 3 \tag{6.237}$$

$$2x_1 + x_2 \geq 4 \tag{6.238}$$

$$x_1, x_2 \geq 0 \tag{6.239}$$

Although we do not need to write down the dual problem to execute the dual simple algorithm, let us do so in this case so that we may track the solutions to both problems graphically. By letting π_1, π_2 designate the dual variables, the dual problem can be written as follows:

$$\text{maximize } 3\pi_1 + 4\pi_2 \tag{6.240}$$

subject to

$$\pi_1 + 2\pi_2 \leq 8 \quad (6.24$$

$$\pi_1 + \pi_2 \leq 5 \quad (6.24$$

$$\pi_1, \pi_2 \geq 0 \quad (6.24$$

Rather than preprocessing the primal problem as usual, instead we shall change the objective to maximization form and multiply both constraints through by -1. Adding slack variables x_3 to constraint (6.241) and x_4 to constraint (6.242) yields

$$\text{maximize } z' = -8x_1 - 5x_2 \quad (6.24$$

subject to

$$-x_1 - x_2 + x_3 = -3 \quad (6.24$$

$$-2x_1 - x_2 + x_4 = -4 \quad (6.24$$

$$\mathbf{x} \geq \mathbf{0} \quad (6.24$$

The initial tableau for the resulting problem is then given in Table 6.9. Note that the initial basis is primal infeasible ($x_3 = -3$ and $x_4 = -4$) and dual feasible (all $z_j - c_j \geq 0$). Thus, the dual simplex algorithm can be employed.

TABLE 6.9

	z'	x_1	x_2	x_3	x_4	RHS
z'	1	8	5	0	0	0
x_3	0	−1	−1	1	0	−3
x_4	0	−2	−1	0	1	−4

STEP 2. The most negative β_i is $\beta_2 = -4$. Thus, $r = 2$ and $x_{B,2} = x_4$ is the departing variable. Go to Step 3.

STEP 3. Because $\alpha_{2,1}$ and $\alpha_{2,2} < 0$, the primal infeasibility condition is not satisfied. Go to Step 4.

STEP 4. We now examine the ratios $(z_j - c_j)/(-\alpha_{2,j})$ where $\alpha_{2,j} < 0$:

$$\frac{z_1 - c_1}{-\alpha_{2,1}} = \frac{8}{-(-2)} = 4$$

$$\frac{z_2 - c_2}{-\alpha_{2,2}} = \frac{5}{-(-1)} = 5$$

Thus, $k = 1$ and the entering variable is x_1.

STEP 5. (a) Because x_1 is the entering variable and x_4 is the departing variable, x_1 replaces x_4 in $\mathbf{x}_B$ as the basic variable in row 2.

(b) Pivot as usual on $\alpha_{2,1} = -2$. This results in Table 6.10.

Note that in Table 6.10, the value of the objective (z') has *decreased*, the top row of the tableau still indicates primal optimality (dual feasibility), and we are not yet primal feasible.

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and x_4 to constraint (6.24

$- 5x_2$

-3

-4

; then given in Table 6.9
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be employed.

x_4	RHS
0	0
0	−3
1	−4

and $x_{B,2} = x_4$ is the d

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$\alpha_{2,j} < 0$:

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TABLE 6.10

	z'	x_1	x_2	x_3	x_4	RHS
z'	1	0	1	0	4	−16
x_3	0	0	$-\frac{1}{2}$	1	$-\frac{1}{2}$	−1
x_1	0	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	2

(c) Return to Step 2.

STEP 2. The only negative β_i is $\beta_1 = -1$. Thus, $r = 1$ and $x_{B,1} = x_3$ is the departing variable. Go to Step 3.

STEP 3. Because $\alpha_{1,2}$ and $\alpha_{1,4} < 0$, the primal infeasibility condition is not satisfied. Go to Step 4.

STEP 4. The ratios $(z_j - c_j)/(-\alpha_{1,j})$, where $\alpha_{1,j} < 0$ are

$$\frac{z_2 - c_2}{-\alpha_{1,2}} = \frac{1}{-(-\frac{1}{2})} = 2$$

$$\frac{z_4 - c_4}{-\alpha_{1,4}} = \frac{4}{-(-\frac{1}{2})} = 8$$

Thus, $k = 2$ and the entering variable is x_2.

STEP 5. (a) x_2 replaces x_3 in $\mathbf{x}_B$ as the basic variable in row 1.
(b) Pivot as usual on $\alpha_{1,2} = -\frac{1}{2}$ to obtain Table 6.11.

Table 6.11 represents the optimal solution because both primal and dual feasibility are satisfied. Note that both the complementary primal and dual solutions can be read from the optimal tableau. These solutions are listed in Table 6.12 (λ_1 and λ_2 are the respective dual slack variables). Observe that complementary slackness is satisfied and that both solutions correspond to an objective value $z^* = -z' = 18$.

TABLE 6.11

	z'	x_1	x_2	x_3	x_4	RHS
z'	1	0	0	2	3	−18
x_2	0	0	1	−2	1	2
x_1	0	1	0	1	−1	1

TABLE 6.12 COMPLEMENTARY PRIMAL AND DUAL SOLUTIONS FOR TABLE 6.11

Primal solution	Complementary dual solution
$x_1 = 1$	$\lambda_1 = 0$
$x_2 = 2$	$\lambda_2 = 0$
$x_3 = 0$	$\pi_1 = 2$
$x_4 = 0$	$\pi_2 = 3$