C&O 463/663 – Convex Optimization and Analysis
Final Examination — Fall 2002

Monday, 9 Dec. 2002, 10:00am – 12.00pm **(2 hours)**

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1

[10:(5,5)]

Consider the constrained convex optimization problem

$$(NLP) \quad \inf f(x) \quad \text{subject to:} \quad g(x) \leq 0, \; x \in X \subset \Re^2,$$

where $f(x) = x_1$, $g(x) = |x_1| + |x_2| - 1$, and the set $X = \{x : \|x\|_\infty \leq 1\}$.

1. Find the set of all optimal solutions and all Lagrange multipliers. (State the details of the Theorem that you used.)
2. Sketch the dual functional.

2

[15:(10,5)]

(**Caratheodory's Theorem for Cones**)
Let K be the cone generated by the subset $S \subset \Re^n$.

1. Show that any nonzero vector $x \in K$ can be represented as a nonnegative combination of no more than n vectors from S,

$$x = \sum_{i=1}^{n} \alpha_i s_i, \text{ for some } \alpha_i \geq 0, \; s_i \in S, \; i = 1, \ldots, n.$$

2. Furthermore, these vectors from S can be chosen to be linearly independent.

3

[20:(5,5,5,5)]

(**Karush-Kuhn-Tucker vectors**)

Consider the convex functions,

$$f, g_1, \cdots, g_m : E \rightarrow (-\infty, +\infty],$$

with $\emptyset \neq \text{dom} f \subseteq \bigcap_i \text{dom} g_i$. Define the convex program

$$\inf\{f(x) | g(x) \leq 0\},$$

where $g \equiv (g_1, g_2, \cdots, g_m)^T$. Also for each $b \in \mathbf{R}^m$, the value function is defined as $v(b) = \inf\{f(x) | g(x) \leq b\}$. Suppose $v(0)$ is finite. We say the vector $\bar{\lambda}$ in $\mathbf{R}_+^m$ is a *Karush-Kuhn-Tucker vector* if it satisfies $v(0) = \inf\{L(x; \bar{\lambda}) |\ x \in E\}$.

Then:

1. Prove that the set of Karush-Kuhn-Tucker vectors is $-\partial v(0)$.

2. Suppose the point $\bar{x}$ is an optimal solution of the convex program. Prove that the set of Karush-Kuhn-Tucker vectors coincides with the set of Lagrangian multiplier vectors for $\bar{x}$.
   ($bar\lambda$ is a Lagrange multiplier vector for $\bar{x}$ if $\bar{x}$ is a critical point of the Lagrangian and complementary slackness holds.)

3. Prove the Slater condition ensures the existence of a KKT vector.

4. Suppose $\bar{\lambda}$ is a Karush-Kuhn-Tucker vector. Prove a feasible point $\bar{x}$ is optimal for the convex program if and only if $\bar{\lambda}$ is a Lagrangian multiplier vector for $\bar{x}$.