C&O 463/663 Convex Optimization and Analysis (Fall 2007) Assignment 4/LAST

Due 1PM (before class), on Thursday, Nov. 29, 2007

1 Conjugate Duality

1. (text page 57) Suppose that $f, g : \mathcal{E} \to (-\infty, +\infty]$ are convex and $(f \odot g)(y) := \inf_x \{f(x) + g(y - x)\}$ is the *infimal convolution*.

   (a) Prove that $f \odot g$ is convex. (On the other hand, if g is concave prove that so is $f \odot g$.)

   (b) Prove $(f \odot g)^* = f^* + g^*$.

2. Generalize and prove the above two problems 1a and 1b to the infimal convolution $h(y) = \inf \left\{ \sum_{i=1}^{k} f_i(x_i) : \sum_{i=1}^{k} x_i = y \right\}$. (Thus the operations + and $\odot$ are dual to each other with respect to taking conjugates.)

2 Linear Cone Duality

1. Consider the problem

$$\begin{array}{ll} \min & \langle c, x \rangle \\ \text{subject to} & \|A_j x + b_j\| \le \langle e_j x \rangle + \alpha_j, \quad j = 1, \dots, r, \end{array}$$

   where $x \in \Re^n$, and c, A_j, b_j, e_j, and α_j are given, and have appropriate dimension. Assume that the problem is feasible. Consider the equivalent problem

$$\begin{array}{ll} \min & \langle c, x \rangle \\ \text{subject to} & \|u_j\| \le t_j, \quad u_j = A_j x + b_j, \quad t_j = \langle e_j, x \rangle + \alpha_j, \quad j = 1, \dots, r, \end{array} \tag{1}$$

   where u_j and t_j are auxiliary optimization variables.

   (a) Show that problem (1) can be written as a linear cone optimization problem.

   (b) Show that a dual problem to (1) can be written as

$$\begin{array}{ll} \min & \sum_{j=1}^{r} \left(\langle b_j^T, z_j \rangle + w_j \alpha_j \right) \\ \text{subject to} & \sum_{j=1}^{r} \left(A_j^T z_j + w_j e_j \right) = c, \quad \|z_j\| \le w_j, \quad j = 1, \dots, r, \end{array}$$

   and provide conditions that guarantee strong duality.

2. (BONUS) Consider the general cone optimization problem

$$\begin{array}{cc} \sup & \langle \mathtt{b}, \mathtt{y} \rangle \\ \text{subject to} & \mathbb{A}^{\mathrm{adj}} \mathtt{y} \preceq_{\mathsf{K}} \mathtt{c}, \end{array} \tag{2}$$

where $\mathbb{A} : \mathcal{E} \to \mathcal{W}$ is a linear transformation between the two finite dimensional Euclidean spaces $\mathcal{E}$ and $\mathcal{W}$. Suppose that there exists $\hat{\mathtt{y}}$ such that $\mathtt{c} - \mathbb{A}^{\mathrm{adj}} \in \operatorname{relint} \mathsf{K}$. Prove that strong duality holds for (2).