C&O 463/663 Convex Optimization and Analysis (Fall 2007)
Assignment 2

Due 1PM (before class), on Tuesday, Oct. 23, 2007

1 Convex, Gauge and Support Functions

1. Suppose that the function $f : E \to (-\infty, +\infty]$ is essentially strictly convex. Prove that all distinct points x, y in E satisfy $\partial f(x) \cap \partial f(y) = \emptyset$. Deduce that f has at most one minimizer.

2. For real $p > 1$, define q by $\frac{1}{p} + \frac{1}{q} = 1$ and for $x \in \mathfrak{R}^n$ define $\|x\|_p = \left(\sum_1^n |x_i|^p\right)^{1/p}$.

   (a) Prove $\frac{1}{p}\|x\|_p^p$ is a convex function and deduce the set $B_p = \{x : \|x\|_p \leq 1\}$ is convex.

   (b) Prove the gauge function $\gamma_{B_p}(\cdot)$ is exactly $\|\cdot\|_p$, and deduce $\|\cdot\|_p$ is convex.

   (c) Use the Fenchel-Young inequality to prove that any vectors x, ϕ in $\mathfrak{R}^n$ satisfy the inequality
   $$\frac{1}{p}\|x\|_p^p + \frac{1}{q}\|\phi\|_q^q \geq \langle \phi, x \rangle.$$

3. Suppose that C, D are closed convex sets in E. Show that $C = D$ if and only if their support functions are equal, $\sigma_C = \sigma_B$.

4. Show that: $\sigma_B = \sigma_{\text{conv } B}$; $\sigma_{A+B} = \sigma_A + \sigma_B$; $\sigma_{A \cup B} = \max\{\sigma_A, \sigma_B\}$.

2 Recession and Normal Cones

1. Suppose that $S \subset E$ is nonempty, closed, and convex. Define the *recession cone of* S to be
   $$0^+S := \{d \in E : x + td \in S, \forall x \in S, \forall t \geq 0\}$$

   (a) Prove that $d \in 0^+S$ iff $\exists x \in S$ s.t. $x + td \in S$, $\forall t \geq 0$.

   (b) Prove that $0^+\text{relint}\,(S) = 0^+\text{cl}\,(S)$, i.e. the recession cones of the relative interior and the closure are equal.

2. (a) Prove that the normal cone is a closed convex cone.

   (b) Compute the normal cone $N_C(\bar{x})$ for points $\bar{x}$ in the sets: $C = B$ the unit ball; C a closed halfspace.