# (Lagrangian) Duality for Linear Programming

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# 1 Preliminary Example

We first consider the following elementary linear program, $\boldsymbol{LP}$ , in Standard Equality Form, $\boldsymbol{SEF}$ .

$$(\boldsymbol{P}) \quad \begin{array}{rcll} \mu^* & := & \max & 3x_1 + 3x_2 - 3x_3 \\ & & \text{subject to} & x_1 + x_2 + 3x_3 = 3 \\ & & & 3x_1 - x_2 - 3x_3 = 7 \\ & & & x \geq 0 \end{array} \tag{1.1}$$

The objective function $c^T x$ has $c^T = \begin{pmatrix} 3 & 3 & -3 \end{pmatrix}$ and the constraints $Ax = b$ have $A = \begin{pmatrix} 1 & 1 & 3 \\ 3 & -1 & -3 \end{pmatrix}$, $b = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$.

> **Exercise:** Verify that $x^* = \begin{pmatrix} \frac{5}{2} & \frac{1}{2} & 0 \end{pmatrix}^T$ is a feasible solution with objective value 9.

The LP $(\boldsymbol{P})$ can be considered to be a production problem, where the manager, called *player* $X$, is looking for a production plan $x \geq 0$ that maximizes profit subject to scheduling and resource constraints.

The purpose of Lagrangian Duality (using Lagrange multipliers, $y$) is to simplify the problem by moving some of the constraints into the objective function. Here we form the Lagrangian by moving the equality constraints into the objective function with Lagrange multipliers $y = (y_i)$.

$$L(x, y) = c^T x + y^T (b - Ax) = c^T x + \sum_{i=1}^{2} y_i (b_i - \sum_{j=1}^{3} A_{ij} x_j) \quad \text{(Lagrangian/payoff)}$$

where $A_{ij}$ denotes the $ij$ element of the matrix $A$. We can also view this in terms of *game theory*, where a player $Y$ is introduced. In this game, $X$ plays strategy $x \geq 0$, while $Y$ plays strategy $y$. The payoff $L(x, y)$ goes from $Y$ to $X$.

## 1.1 Best Strategy for Player $X$

What is the best strategy for player $X$ over all possible strategies for player $Y$?

$$p_X^* := \max_{x \geq 0} \min_{y} L(x, y) = \max_{x \geq 0} \left( \min_{y} L(x, y) \right), \tag{1.2}$$

i.e. player $X$ wants to maximize the payoff to him over all possible strategies for player $Y$. However, we now see that there is a *hidden constraint* for player $X$. He cannot allow $b - Ax \neq 0$. If he did, then $Y$ may play a $y$ that makes $y^T (b - Ax)$ unboundedly negative. For example, suppose that $X$ plays $x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Then $b - Ax = \begin{pmatrix} -2 \\ 8 \end{pmatrix}$, and $y^T (b - Ax) = -2y_1 + 8y_2$. The latter value can be made arbitrarily negative by choosing $y$ with $y_1$ large positive and $y_2$ very negative.

Therefore, we conclude that the best strategy for $X$ becomes

$$p_X^* = \max_{Ax=b, x\geq 0} \min_y L(x,y) = c^T x + y^T(b - Ax). \tag{1.3}$$

We see that the inner minimization problem makes

$$y^T(b - Ax) = 0 \qquad \text{(called complementary slackness)} \tag{1.4}$$

and then the inner minimization becomes redundant, i.e. we have shown that $p_X^* = \mu^*$ and the optimal strategy for player $X$ is obtained from solving $(\boldsymbol{P})$ , the primal $\boldsymbol{LP}$ .

## 1.2 Best Strategy for Player $Y$

What is the best strategy for player $Y$ over all possible strategies for player $X$? We interchange the max and the min. (Note that maximizing first results in a larger value.) We get, after taking transposes and combining $x$ terms,

$$\mu^* = p_X^* \leq p_Y^* := \min_y \max_{x\geq 0} L(x,y) = c^T x + y^T(b - Ax) = b^T y + x^T(c - A^T y), \tag{1.5}$$

i.e. player $Y$ wants to minimize the payoff that she pays to $X$ over all possible strategies that player $X$ can use. This min-max problem is the *Lagrangian Dual* of $(\boldsymbol{P})$ . The inequality in (1.5) is called *Weak Duality*, i.e.

> **WEAK DUALITY:** *The optimal dual value $p_Y^*$ of the minimization problem* $(\boldsymbol{D})$ *provides an upper bound for the optimal primal value $p_X^* = \mu^*$ of the maximization problem* $(\boldsymbol{P})$ *, i.e. $p_X^* \leq p_Y^*$.* ■

Note that this statement of weak duality includes the cases where $p_X^* = +\infty = p_Y^*$ or $-\infty = p_X^* \leq p_Y^* = +\infty$. (The statement of weak duality implies **Theorem 4.1** in the class notes.)

We now see from (1.5) that player $Y$ also has a *hidden constraint*, which allows us to simplify the dual problem. Player $Y$ cannot allow $x^T(c - A^T y)$ to become unboundedly large. Since she has no control over which $x \geq 0$ is played by $X$, her hidden constraint is $c - A^T y \leq 0$. Therefore, we get

$$p_Y^* := \min_{A^T y \geq c} \max_{x\geq 0} L(x,y) = b^T y + x^T(c - A^T y). \tag{1.6}$$

We see that the inner maximization problem makes

$$x^T(c - A^T y) = 0 \qquad \text{(called complementary slackness)} \tag{1.7}$$

and the inner maximization problem becomes redundant, i.e. we have shown $p_Y^*$ is obtained from a simplified Lagrangian dual, the ***LP*** dual,

$$(\boldsymbol{D}) \qquad \begin{array}{rcll} p_Y^* & = & \min & 3y_1 + 7y_2 \\ & & \text{subject to} & y_1 + 3y_2 \geq 3 \\ & & & y_1 - y_2 \geq 3 \\ & & & 3y_1 - 3y_2 \geq -3 \end{array} \tag{1.8}$$

**Exercise:** Verify that $y^* = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ is feasible for $(\boldsymbol{D})$ with objective value 9. Thus we have proved that $y^*$ is optimal for $(\boldsymbol{D})$ and $x^* = \frac{1}{2}\begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}$ given above is optimal for $(\boldsymbol{P})$ . Why?

## 1.3 Best Upper Bound and Alternate Duals

The Lagrangian dual, as seen in Player $Y$'s strategy, can also be derived from trying to obtain a best upper bound for the optimal value of $(\boldsymbol{P})$ . Since we add the equality constraints into the objective function, we see that, for any fixed strategy $y$, the inner maximization problem in (1.5) provides an upper bound for the primal optimal value

$$\mu^* \leq \max_{x \geq 0} L(x, y), \text{ for any } y. \tag{1.9}$$

To obtain the best upper bound, we find the smallest or minimum bound and obtain the result in (1.5). This then reduces to the simplified dual in $(\boldsymbol{D})$ , since the inner maximization in (1.9) has value $+\infty$ when the dual constraint $c - A^T y \leq 0$ fails.

However, one can now ask why only add the equality constraints to the objective function? Why not add some or all of the inequalities in $x \geq 0$? Suppose we add all the constraints. Then we get the Lagrangian

$$L(x, y, z) = c^T x + y^T(b - Ax) + z^T x, \qquad z \leq 0.$$

The best strategy for player $X$ is

$$p_X^* := \max_x \min_{z \leq 0, y} L(x,y,z) = c^T x + y^T(b - Ax) + z^T x \qquad (1.10)$$

Player $X$ has the two hidden constraints $Ax = b, x \geq 0$ that prevent the payoff from becoming unboundedly negative. Therefore, $p_X^* = \mu^*$ again, i.e. the best strategy is obtained from $\boldsymbol{(P)}$ again. (Thus we see why we had to restrict $z \leq 0$.)

The best strategy for player $Y$ is

$$\begin{aligned} p_Y^* := \min_{z \leq 0, y} \max_x L(x,y,z) &= c^T x + y^T(b - Ax) + z^T x \\ &= b^T y + 0z + x^T \left( c - A^T y + z \right) \end{aligned} \qquad (1.11)$$

The hidden constraint for player $Y$, since $x$ is free, is $( c - A^T y + z ) = 0$. We therefore get the dual program with equality and nonnegativity constraints

$$\boldsymbol{(D)} \quad \begin{array}{rcll} p_Y^* & = & \min & 3y_1 + 7y_2 \\ & & \text{subject to} & y_1 + 3y_2 + z_1 = 2 \\ & & & y_1 - y_2 + z_2 = 3 \\ & & & 3y_1 - 3y_2 + z_3 = -1 \\ & & & z \leq 0 \end{array} \qquad (1.12)$$

Which is the correct dual, (1.8) or (1.12)? Clearly, there are many other choices depending on which constraints we choose to add to the objective function. In the literature and the course notes, the dual of choice for $\boldsymbol{(P)}$ is (1.8). For some practical applications, (1.12) is used since the dual variables $z$ is needed.

# 2 Primal-Dual Pairs

From the above we have shown that the primal $\boldsymbol{LP}$ in $\boldsymbol{SEF}$

$$\boldsymbol{(P)} \quad \begin{array}{rcll} \mu^* & := & \max & c^T x \\ & & \text{subject to} & Ax = b \\ & & & x \geq 0 \end{array}$$

has the dual $\boldsymbol{LP}$ problem

$$\boldsymbol{(D)} \quad \begin{array}{rcll} \nu^* & := & \min & b^T y \\ & & \text{subject to} & A^T y \geq c \end{array}$$

But, what about other types of $\boldsymbol{LP}$s? We consider the general $\boldsymbol{LP}$

$$(\boldsymbol{P}) \qquad \begin{array}{rlc} \mu^* := & \max & c_1^T x_1 + c_2^T x_2 + c_3^T x_3 \\ & \text{subject to} & A_{11}x_1 + A_{12}x_2 + A_{13}x_3 = b_1 \\ & & A_{21}x_1 + A_{22}x_2 + A_{23}x_3 \le b_2 \\ & & A_{31}x_1 + A_{32}x_2 + A_{33}x_3 \ge b_3 \\ & & x_1 \ge 0, x_2 \le 0, x_3 \text{ free} \\ & & x_1 \in \Re^{n_1}, x_2 \in \Re^{n_2}, x_3 \in \Re^{n_3}, \end{array} \tag{2.1}$$

where the $x_1, x_2, x_3$ are vectors and the data $c_j, A_{ij}, b_i$ are matrices/vectors of appropriate sizes. If we consider the constraints as resource constraints and introduce player $Y$ who puts a value on the resources, then player $Y$ plays $y_1 \in \Re^{m_2}, y_2 \in \Re^{m_1}, y_3 \in \Re^{m_3}$ with

$$y_1 \text{ free}, \; y_2 \ge 0, y_3 \le 0.$$

The reasoning is that an increase in $b_3$ results in a smaller feasible set and a decrease in the profit $\mu^*$, while an increase in $b_2$ results in a larger feasible set and an increase in the profit $\mu^*$. Changes in $b_1$ can result in both an increase or a decrease in profit.

Therefore, the Lagrangian/payoff function is

$$L(x_1, x_2, x_3, y_1, y_2, y_3) = \sum_{j=1}^{3} c_j^T x_j + \boxed{\sum_{i=1}^3 y_i^T (b_i - \sum_{j=1}^3 A_{ij} x_j)} \tag{2.2}$$

The optimal strategy for player $X$ is

$$p_X^* := \max_{\substack{x_1 \ge 0, x_2 \le 0 \\ x_3}} \min_{\substack{y_2 \ge 0, y_3 \le 0 \\ y_1}} L(x_1, x_2, x_3, y_1, y_2, y_3) \tag{2.3}$$

As above, the optimal strategy for player $Y$ is obtained by interchanging max and min, i.e.

$$p_X^* \le p_Y^* = \min_{\substack{y_2 \ge 0, y_3 \le 0 \\ y_1}} \max_{\substack{x_1 \ge 0, x_2 \le 0 \\ x_3}} L(x_1, x_2, x_3, y_1, y_2, y_3). \tag{2.4}$$

Weak duality follows from the inequality.

We then rewrite the Lagrangian by taking transposes and bringing the terms with $x$ together.

$$L(x_1, x_2, x_3, y_1, y_2, y_3) = \sum_{i=1}^{3} b_j^T y_j + \boxed{\sum_{j=1}^3 x_j^T (c_j - \sum_{i=1}^3 A_{ji}^T y_i)} \tag{2.5}$$

For $x_1 \geq 0$, we get the hidden constraint $(c_1 - \sum_{i=1}^3 A_{1i}^T y_i) \leq 0$. For $x_2 \leq 0$, we get the hidden constraint $(c_2 - \sum_{i=1}^3 A_{2i}^T y_i) \geq 0$. For $x_3$ free, we get the hidden constraint $(c_3 - \sum_{i=1}^3 A_{3i}^T y_i) = 0$. The inner maximization becomes redundant and we get the dual

$$
\textbf{(D)} \qquad
\begin{array}{rll}
\nu^* := & \min & b_1^T y_1 + b_2^T y_2 + b_3^T y_3 \\
& \text{subject to} & A_{11}^T y_1 + A_{21}^T y_2 + A_{31}^T y_3 \geq c_1 \\
& & A_{12}^T y_1 + A_{22}^T y_2 + A_{32}^T y_3 \leq c_2 \\
& & A_{13}^T y_1 + A_{23}^T y_2 + A_{33}^T y_3 = c_3 \\
& & y_1 \text{ free }, y_2 \geq 0, y_3 \leq 0.
\end{array}
\qquad (2.6)
$$

# 3 Complementary Slackness

Suppose that $x_1, x_2, x_3$ is feasible for **(P)** in (2.1) and that $y_1, y_2, y_3$ is feasible for **(D)** in (2.6). Then the primal objective value $\sum_{j=1}^3 c_j x_j = p_X^*$ in (2.3) if and only if the value emphasized in (2.2) $\boxed{\sum_{i=1}^3 y_i^T (b_i - \sum_{j=1}^3 A_{ij} x_j) = 0}$. Similarly, the dual objective value $\sum_{i=1}^3 b_j y_j = p_Y^*$ in (2.4) if and only if the value emphasized in (2.5) $\boxed{\sum_{j=1}^3 x_j^T (c_j - \sum_{i=1}^3 A_{ji}^T y_i) = 0}$. Since the signs in the summations are the same, this leads to two equivalent forms of complementary slackness.

> **Complementary Slackness** Suppose that $x_1, x_2, x_3$ is feasible for **(P)** in (2.1) and that $y_1, y_2, y_3$ is feasible for **(D)** in (2.6).
> **Then:**
>
> $x_1, x_2, x_3$ is optimal for **(P)** in (2.1) and $y_1, y_2, y_3$ is optimal for **(D)** in (2.6)
>
> <u>**if and only if**</u>
>
> $$\boxed{\sum_{i=1}^3 y_i^T (b_i - \sum_{j=1}^3 A_{ij} x_j) = 0}.$$
>
> and
>
> $$\boxed{\sum_{j=1}^3 x_j^T (c_j - \sum_{i=1}^3 A_{ji}^T y_i) = 0}.$$
>
> <u>**if and only if**</u>

$$\boxed{y_i^T = 0 \text{ or } (b_i - \sum_{j=1}^{3} A_{ij}x_j) = 0, \quad \text{for each } i = 1,2,3}.$$

and

$$\boxed{x_j^T = 0 \text{ or } (c_j - \sum_{i=1}^{3} A_{ji}^T y_i) = 0, \quad \text{for each } j = 1,2,3}.$$

■

**Exercise:** For the first example (1.1) and its dual, use complementary slackness to verify that $x^* = \frac{1}{2}\begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}$, $y^* = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ is an optimal primal-dual pair.

# 4 Historical Notes

When George Dantzig, the *father of Linear Programming*, had proposed the linear programming model in 1947, he consulted with John von Neumann at the Institute for Advanced Study at Princeton. He quickly discovered that Von Neumann was well aware of linear programming, as a result of a book on the *Theory of Games* that he had just completed with Oscar Morgenstern. It was this way that Dantzig learned about *Duality* as well as about Farkas' Lemma (to be shown later in the course). For more details on this meeting and Dantzig's reminiscences see [1, Page 24].

# References

[1] J.K. LENSTRA, A.H.G. RINNOY KAN, and A. SCHRIJVER. *History of Mathematical Programming: A Collection of Personal Reminiscences.* CWI North-Holland, Amsterdam, 1991.