CO350 Linear Programming Chapter 9: The Revised Simplex Method

11th July 2005

Example of infeasibility

Solve the LP using revised two-phase method with smallest-subscript rules.

$$
(P)\quad
\begin{array}{lrl}
\text{max} & (z =) & 2x_1 - x_2 \\
\text{s.t.} & & 3x_1 + 10x_2 + 7x_3 + 4x_4 = 2 \\
 & & 2x_1 + 5x_2 + 3x_3 + x_4 = 3 \\
 & & x_1 , x_2 , x_3 , x_4 \geq 0
\end{array}
$$

The auxiliary problem is

$$
(P')\quad
\begin{array}{lrcl}
\text{max} & g^T x & & \\
\text{s.t.} & Dx & = & f \\
 & x & \geq & 0
\end{array}
$$

where

$$
D = \begin{bmatrix} 3 & 10 & 7 & 4 & 1 & 0 \\ 2 & 5 & 3 & 1 & 0 & 1 \end{bmatrix}, \quad f = \begin{bmatrix} 2 \\ 3 \end{bmatrix}
$$

and

$$
g = [0, 0, 0, 0 - 1, -1]^T
$$

Iteration 1:

$B = \{5, 6\}$, $x^*_B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $A_B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Solve $A^T_B y = c_B = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ to get $y = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$.

Compute $\bar{c}_1 = c_1 - A^T_1 y = 0 - [\ 3 \quad 2\ ]y = 5 > 0$. x_1 enters.

Solve $A_B d = A_1$ to get $d = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

$t = \min\{\frac{2}{3}, \frac{3}{2}\} = \frac{2}{3}$. x_5 leaves.

Update $x^*_1 = t = \frac{2}{3}$, $x^*_6 = 3 - (2)(\frac{2}{3}) = \frac{5}{3}$.

Drop artificial x_5.

Iteration 2:

$B = \{1, 6\}$, $x^*_B = \begin{bmatrix} 2/3 \\ 5/3 \end{bmatrix}$, $A_B = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}$.

Solve $A^T_B y = c_B = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ to get $y = \begin{bmatrix} 2/3 \\ -1 \end{bmatrix}$.

Compute $\bar{c}_2 = c_2 - A^T_2 y = 0 - [\ 10 \quad 5\ ]y = -5/3 \leq 0$.

Compute $\bar{c}_3 = c_3 - A^T_3 y = 0 - [\ 7 \quad 3\ ]y = -5/3 \leq 0$.

Compute $\bar{c}_4 = c_4 - A^T_4 y = 0 - [\ 4 \quad 1\ ]y = -5/3 \leq 0$.

An optimal solution of (A) is $x^* = [2/3, 0, 0, 0, 0, 5/3]^T$ with optimal value $-5/3 < 0$.

Thus (P) is infeasible.

Proof of infeasibility

$(\frac{2}{3}) \times$ (eqn. $1) + (-1) \times$ (eqn. 2):

$$\frac{5}{3}x_2 + \frac{5}{3}x_3 + \frac{5}{3}x_4 = -\frac{5}{3}$$

$\hat{x}$ is feasible $\implies 0 \leq \frac{5}{3}\hat{x}_2 + \frac{5}{3}\hat{x}_3 + \frac{5}{3}\hat{x}_4 = -\frac{5}{3}$. (Contradiction)

Remarks on Implementation (§9.4)

Updating of tableau in simplex method is replaced by solving two system of equations involving A_B in revised simplex method.

It is not immediately clear that the revised simplex method is better.

To see it, we need to study the details of implementation and typical properties of large LP problems.

(1) **Basis factorization.**

We solve $A_B^T y = c_B$ and $A_B d = A_k$ at each iteration.

This is usually done by factorizing A_B.

A_B changes by only one column at each iteration.

Instead of re-factorizing A_B, it is less work to add a new factor that reflects the one-column change.

However, we should re-factorize periodically to keep the number of factors in check.

(2) **Entering variable rules.**

By computational experiments, some choice rules are found to reduce the number of iterations.

There is no useful theoretical result that puts one choice rule above the others.

(3) **Problem shape.**

When n is much larger than m, the revised simplex method is usually faster.

When choosing entering variable, we may need to compute all $(n-m)$ $\bar{c}_j$.

Solution: partial pricing – computing reduced costs only for a selected subset of nonbasic variables.

(4) **Sparsity.**

Sparsity refers to the (low) proportion of nonzero coefficients in A.

A sparse A usually means that the revised simplex method is better.

Reason: $\bar{A}$ may be very dense while A_B is usually Sparse.

CO350 Linear Programming
Chapter 10: Sensitivity Analysis

11th July 2005

Introduction

This chapter deals with post-optimality changes to LP problems;
i.e., changes made to the data after obtaining optimal solution and basis.

Why do we study such changes?

1. LP problems arising from similar situations have similar data.

2. Data for an LP problem may not be precise, but fall in a (small) range.

3. Smaller LP problems (a.k.a. subproblems) used as modules to solve much larger problems usually have similar data.

Given:

- an optimal basis B of an LP problem (P),
- an LP problem (P') that differs slightly from (P).

Two types of questions

1. Is B also optimal for (P')?
   If not, how can we exploit B in solving (P')?

2. How much can (P') differ before the optimal basis B is no longer optimal for it?

Example (from 29th June)

$$
(P) \qquad
\begin{array}{lrcrcrcrcrcrcl}
\max & -x_1 & + & x_2 & - & x_3 & - & x_4 & - & x_5 & + & x_6 & & \\
\text{s.t.} & -8x_1 & - & 3x_2 & + & 12x_3 & + & x_4 & & & & & = & 3 \\
 & -2x_1 & - & x_2 & + & 6x_3 & & & + & x_5 & & & = & 4 \\
 & 3x_1 & + & x_2 & - & 4x_3 & & & & & + & x_6 & = & 2 \\
 & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & , & x_6 & \geq & 0
\end{array}
$$

After solving (P):

$$
\begin{array}{rcl}
\text{optimal basis} & — & B = \{2, 3, 6\} \\
\text{optimal solution} & — & x^* = [0, 5, \frac{3}{2}, 0, 0, 3]^T \\
\text{optimal dual solution} & — & y^* = [-\frac{1}{2}, \frac{3}{2}, 1]^T
\end{array}
$$

Note: x^* solves $A_B x_B = b$, and
y^* solves $A_B^T y = c_B$.

Consider the following five types of changes.

1. Changes in nonbasic c_j.
2. Addition of new variable.
3. Changes in basic c_j.
4. Changes in b_i.
5. Addition of new constraint.