CO350 Linear Programming Chapter 9: The Revised Simplex Method

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Example of unboundedness

Solve the LP using revised simplex method with smallest-subscript rules.

$$
(P) \quad
\begin{array}{lrcrcrcl}
\max & 3x_1 & - & 2x_2 & - & 3x_3 & & \\
\text{s.t.} & x_1 & - & x_2 & - & x_3 & \leq & 1 \\
 & 7x_1 & - & 8x_2 & - & 11x_3 & \leq & 2 \\
 & 2x_1 & - & 2x_2 & - & 3x_3 & \leq & 1 \\
 & x_1 & , & x_2 & , & x_3 & \geq & 0
\end{array}
$$

Adding slack variables x_4, x_5, x_6 gives

$$
(P') \quad
\begin{array}{lrcl}
\max & c^T x & & \\
\text{s.t.} & Ax & = & b \\
 & x & \geq & 0
\end{array}
$$

where

$$
A = \begin{bmatrix} 1 & -1 & -1 & 1 & 0 & 0 \\ 7 & -8 & -11 & 0 & 1 & 0 \\ 2 & -2 & -3 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}
$$

and

$$
c = [3, -2, -3, 0, 0, 0]^T
$$

Start from feasible basis $B = \{4, 5, 6\}$.

Iteration 1:

$$B = \{4,5,6\},\ x_B^* = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix},\ A_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solve $A_B^T y = c_B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ to get $y = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Compute $\bar{c}_1 = c_1 - A_1^T y = 3 - [\ 1 \quad 7 \quad 2\ ]y = 3 > 0.$

x_1 enters.

Solve $A_B d = A_1$ to get $d = \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}$.

$t = \min\{\frac{1}{1}, \frac{2}{7}, \frac{1}{2}\} = \frac{2}{7}$. x_5 leaves.

Update $x_1^* = t = \frac{2}{7}$, $x_4^* = 1 - (1)(\frac{2}{7}) = \frac{5}{7}$, $x_6^* = 1 - (2)(\frac{2}{7}) = \frac{3}{7}$.

Iteration 2:

$$B = \{1,4,6\},\ x_B^* = \begin{bmatrix} 2/7 \\ 5/7 \\ 3/7 \end{bmatrix},\ A_B = \begin{bmatrix} 1 & 1 & 0 \\ 7 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

Solve $A_B^T y = c_B = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$ to get $y = \begin{bmatrix} 0 \\ 3/7 \\ 0 \end{bmatrix}$.

Compute $\bar{c}_2 = c_2 - A_2^T y = -2 - [\ -1 \quad -8 \quad -2\ ]y = 10/7 > 0$.
x_2 enters.

Solve $A_B d = A_2$ to get $d = \begin{bmatrix} -8/7 \\ 1/7 \\ 2/7 \end{bmatrix}$.

$t = \min\{\text{-}, \frac{5/7}{1/7}, \frac{3/7}{2/7}\} = \frac{3}{2}$. x_6 leaves.
Update $x_2^* = t = \frac{3}{2}$, $x_1^* = \frac{2}{7} - (-\frac{8}{7})(\frac{3}{2}) = 2$, $x_4^* = \frac{5}{7} - (\frac{1}{7})(\frac{3}{2}) = \frac{1}{2}$.

Iteration 3:

$B = \{1, 2, 4\}$, $x_B^* = \begin{bmatrix} 2 \\ 3/2 \\ 1/2 \end{bmatrix}$, $A_B = \begin{bmatrix} 1 & -1 & 1 \\ 7 & -8 & 0 \\ 2 & -2 & 0 \end{bmatrix}$.

Solve $A_B^T y = c_B = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}$ to get $y = \begin{bmatrix} 0 \\ -1 \\ 5 \end{bmatrix}$.

Compute $\bar{c}_3 = c_3 - A_3^T y = -3 - [\ -1 \quad -11 \quad -3\ ]y = 1 > 0$.
x_3 enters.

Solve $A_B d = A_3$ to get $d = \begin{bmatrix} -1 \\ 1/2 \\ 1/2 \end{bmatrix}$.

$t = \min\{\text{-}, \frac{3/2}{1/2}, \frac{1/2}{1/2}\} = 1$. x_4 leaves.
Update $x_3^* = t = 1$, $x_1^* = 2 - (-1)(1) = 3$, $x_2^* = \frac{3}{2} - (\frac{1}{2})(1) = 1$.

<u>Iteration 4:</u>

$$B = \{1, 2, 3\},\ x_B^* = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix},\ A_B = \begin{bmatrix} 1 & -1 & -1 \\ 7 & -8 & -11 \\ 2 & -2 & -3 \end{bmatrix}.$$

Solve $A_B^T y = c_B = \begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix}$ to get $y = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$.

Compute $\bar{c}_4 = c_4 - A_4^T y = 0 - [\ 1 \quad 0 \quad 0\ ]y = -2 \leq 0$.

Compute $\bar{c}_5 = c_5 - A_5^T y = 0 - [\ 0 \quad 1 \quad 0\ ]y = 1 > 0$.

x_5 enters.

Solve $A_B d = A_5$ to get $d = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$.

$d \leq 0 \implies (P')$ is unbounded.

Hence (P) is unbounded.

<u>Proof of unboundedness</u>

Let $x_5(t) = t$, $x_4(t) = x_6(t) = 0$, and $x_B(t) = x_B^* - td$

i.e., $[x_1(t), x_2(t), x_3(t)]^T = [3 + t, 1 + t, 1]^T$

Check: $x(t)$ is feasible for (P') when $t \geq 0$.

Objective value: $c^T x(t) = 4 + t \to \infty$ as $t \to \infty$

Example of revised two-phase method (see also §9.3)

Solve the LP using revised two-phase method with smallest-subscript rules.

$$(P)\quad \begin{array}{lrl} \max & (z =) & x_1 + 3x_3 \\ \text{s.t.} & & x_1 + 3x_2 - x_3 + 2x_4 = 5 \\ & & x_1 - 3x_2 + 5x_3 - 4x_4 = -1 \\ & & x_1, \; x_2, \; x_3, \; x_4 \geq 0 \end{array}$$

Multiply the second equation by -1 and add artificial variables x_5, x_6 to get the auxiliary problem

$$(A)\quad \begin{array}{lrcl} \max & g^T x & & \\ \text{s.t.} & Dx & = & f \\ & x & \geq & 0 \end{array}$$

where

$$D = \begin{bmatrix} 1 & 3 & -1 & 2 & 1 & 0 \\ -1 & 3 & -5 & 4 & 0 & 1 \end{bmatrix}, \quad f = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

and

$$g = [0, 0, 0, 0 - 1, -1]^T$$

Iteration 1:

$B = \{5, 6\}$, $x_B^* = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$, $D_B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Solve $D_B^T y = g_B = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ to get $y = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$.

Compute $\bar{g}_1 = g_1 - D_1^T y = 0 - [\ 1 \quad -1\ ]y = 0 \leq 0$.

Compute $\bar{g}_2 = g_2 - D_2^T y = 0 - [\ 3 \quad 3\ ]y = 6 > 0$. x_2 enters.

Solve $D_B d = D_2$ to get $d = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$.

$t = \min\{\frac{5}{3}, \frac{1}{3}\} = \frac{1}{3}$. x_6 leaves. Drop artificial x_6.
Update $x_2^* = t = \frac{1}{3}$, $x_5^* = 5 - (3)(\frac{1}{3}) = 4$.

Iteration 2:

$B = \{2, 5\}$, $x_B^* = \begin{bmatrix} 1/3 \\ 4 \end{bmatrix}$, $D_B = \begin{bmatrix} 3 & 1 \\ 3 & 0 \end{bmatrix}$.

Solve $D_B^T y = g_B = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ to get $y = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Compute $\bar{g}_1 = g_1 - D_1^T y = 0 - [\ 1 \quad -1\ ]y = 2 > 0$. x_1 enters.

Solve $D_B d = D_1$ to get $d = \begin{bmatrix} -1/3 \\ 2 \end{bmatrix}$.

$t = \min\{\text{-}, \frac{4}{2}\} = \frac{4}{2}$. x_5 leaves. Drop artificial x_5.
Update $x_1^* = t = 2$, $x_2^* = \frac{1}{3} - (-\frac{1}{3})(2) = 1$.

All artificial variables are dropped, hence $B = \{1, 2\}$ is feasible for (P).

We start Phase 2 on (P) with $B = \{1, 2\}$.

Iteration 1:

$B = \{1, 2\}$, $x^*_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $A_B = \begin{bmatrix} 1 & 3 \\ 1 & -3 \end{bmatrix}$.

Solve $A_B^T y = c_B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to get $y = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$.

Compute $\bar{c}_3 = c_3 - A_3^T y = 3 - [\ -1 \quad 5\ ] y = 1 > 0$. x_3 enters.

Solve $A_B d = A_3$ to get $d = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

$t = \min\{\frac{2}{2}, \text{-}\} = \frac{2}{2}$. x_1 leaves.
Update $x^*_3 = t = 1$, $x^*_2 = 1 - (-1)(1) = 2$.

Iteration 2:

$B = \{2,3\}$, $x_B^* = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $A_B = \begin{bmatrix} 3 & -1 \\ -3 & 5 \end{bmatrix}$.

Solve $A_B^T y = c_B = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ to get $y = \begin{bmatrix} 3/4 \\ 3/4 \end{bmatrix}$.

Compute $\bar{c}_1 = c_1 - A_1^T y = 1 - [\ 1 \quad 1\ ]y = -1/2 \leq 0$.

Compute $\bar{c}_4 = c_4 - A_4^T y = 0 - [\ 2 \quad -4\ ]y = 3/2 > 0$. x_4 enters.

Solve $A_B d = A_4$ to get $d = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}$.

$t = \min\{\frac{2}{1/2}, \text{-}\} = \frac{2}{1/2}$. x_2 leaves.

Update $x_4^* = t = 4$, $x_3^* = 1 - (-\frac{1}{2})(4) = 3$.

Iteration 3:

$B = \{3,4\}$, $x_B^* = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $A_B = \begin{bmatrix} -1 & 2 \\ 5 & -4 \end{bmatrix}$.

Solve $A_B^T y = c_B = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ to get $y = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Compute $\bar{c}_1 = c_1 - A_1^T y = 1 - [\ 1 \quad 1\ ]y = -2 \leq 0$.

Compute $\bar{c}_2 = c_2 - A_2^T y = 0 - [\ 3 \quad -3\ ]y = -3 \leq 0$.

An optimal solution is $x^* = [0, 0, 3, 4]^T$.