CO350 Linear Programming Chapter 9: The Revised Simplex Method

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Motivation

The most computationally intensive part of the simplex method is pivoting;
i.e., the construction of the new tableau.

Tableau:

$$\begin{array}{rcrcll} z & - & \sum_{j \in N} \bar{c}_j x_j & = & \bar{v} & \\ x_i & + & \sum_{j \in N} \bar{a}_{ij} x_j & = & \bar{b}_i & (i \in B) \end{array}$$

In a tableau, there are

- $n - m$ nonbasic $\bar{c}_j$'s
- $(n-m) \times m$ $\bar{a}_{ij}$'s,
- one $\bar{v}$,
- m $\bar{b}_i$'s.

In all, there are $(m+1)(n-m+1)$ numbers in a tableau.

However, we only need

- $n - m$ nonbasic $\bar{c}_j$'s to decide entering variable x_k, and
- m $\bar{a}_{ik}$'s and m $\bar{b}_i$'s to decide leaving variable x_r.

In all, we only need $n + m$ numbers.

Revised Simplex Method

An implementation of the simplex method that computes only the necessary coefficients instead of the whole tableau.

Entering variable

We need to compute nonbasic $\bar{c}_j$.

By definition, a tableau is derived from

$$\begin{aligned} z \quad - \quad c^T x &= 0 \\ Ax &= b \end{aligned}$$

using elementary row operations.

Thus there exists $y = [y_1, y_2, \ldots, y_m]^T$ such that the z-row

$$z - \sum_{j \in N} \bar{c}_j x_j = \bar{v}$$

is equivalent to the linear combination

$$z - c^T x + y^T A x = y^T b;$$

i.e.

$$z - (c - A^T y)^T x = y^T b$$

Comparing coefficients of basic x_i:

$$-c_i + A_i^T y = 0 \qquad \text{for } i \in B$$

i.e.

$$A_B^T y = c_B$$

Comparing coefficients of nonbasic x_j:

$$-c_j + A_j^T y = -\bar{c}_j \qquad \text{for } j \in N$$

i.e.

$$\bar{c}_j = c_j - A_j^T y \qquad \text{for } j \in N$$

To compute $\bar{c}_j$:

1. Solve for y in $A_B^T y = c_B$.
2. Compute $\bar{c}_j = c_j - A_j^T y$.

Leaving variable

Suppose x_k is entering variable.

We need to compute $\bar{a}_{ik}$ and $\bar{b}_i$ for all $i \in B$.

Computing $\bar{a}_{ik}$

From $Ax = b$, pre-multiplying by A_B^{-1} gives

$$A_B^{-1}Ax = A_B^{-1}b$$

The matrix of basic columns in $A_B^{-1}A$ is $A_B^{-1}A_B = I$.

So $A_B^{-1}Ax = A_B^{-1}b$ gives the x_i-rows in the tableau.

Hence, the column for x_k in tableau is $A_B^{-1}A_k$.

To compute $\bar{a}_{ik}$ for all $i \in B$:

1. Solve for d in $A_B d = A_k$.
2. $\bar{a}_{ik} = d_h$, where i is the h-th index in B.

Computing $\bar{b}_i$

$\bar{b}_i$ is the value of x_i^* in the current basic solution x^*.

So we only need to update the basic variables.

Recall:

To compute new basic variables:

new $x_k^* = t =$ min. ratio

new $x_i^* =$ (old x_i^*) $- a_{ik}t$ for $i \in B$, $i \neq r$.

Revised Simplex Method (pg 123)

1. Start with feasible basis B and b.f.s. x^*.
   [If x^* is not given, compute it by solving $A_B x_B = b$]

2. Solve for y in $A_B^T y = c_B$.
   [Remember the transpose in A_B^T]

3. Compute $\bar{c}_j = c_j - A_j^T y$ for each $j \in N$.
   (Optimality) If $\bar{c}_j \leq 0$ for $j \in N$, stop.
   (Entering variable) Pick k with $\bar{c}_k > 0$ using choice rule.

4. Solve for d in $A_B d = A_k$.
   [We label the entries in d using basic indices.]

5. (Unboundedness) If $d \leq 0$, stop.

6. Compute min. ratio: $t = \min\{x_i^*/d_i : i \in B,\ d_i > 0\}$.
   [Remember: entries in d are labelled using basic indices.]
   (Leaving variable) Pick r with $d_r > 0$ and $x_r^*/d_r = t$ using choice rule.

7. (Update basic variables) Replace x_k^* by t and x_i^* by $x_i^* - \bar{a}_{ik} t = x_i^* - d_i t$.

8. (Update basis) Replace B by $(B \cup \{k\}) \setminus \{r\}$.
   Go to step 2.

Example (see also §9.2)

Solve the LP using revised simplex method with smallest-subscript rules starting from the basis $B = \{4, 5, 6\}$.

$$
\begin{array}{lrcrcrcrcrcrcl}
\text{max} & -x_1 & + & x_2 & - & x_3 & - & x_4 & - & x_5 & + & x_6 & & \\
\text{s.t.} & -8x_1 & - & 3x_2 & + & 12x_3 & + & x_4 & & & & & = & 3 \\
 & -2x_1 & - & x_2 & + & 6x_3 & & & + & x_5 & & & = & 4 \\
 & 3x_1 & + & x_2 & - & 4x_3 & & & & & + & x_6 & = & 2 \\
 & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & , & x_6 & \geq & 0
\end{array}
$$

Iteration 1:

$B = \{4, 5, 6\}$, $x_B^* = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$, $A_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Solve $A_B^T y = c_B = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ to get $y = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

Compute $\bar{c}_1 = c_1 - A_1^T y = -1 - [\ -8 \quad -2 \quad 3\ ] y = -14 \leq 0$.

Compute $\bar{c}_2 = c_2 - A_2^T y = 1 - [\ -3 \quad -1 \quad 1\ ] y = -4 \leq 0$.

Compute $\bar{c}_3 = c_3 - A_3^T y = -1 - [\ 12 \quad 6 \quad -4\ ] y = 21 > 0$.
x_3 enters.

Solve $A_B d = A_3$ to get $d = \begin{bmatrix} 12 \\ 6 \\ -4 \end{bmatrix}$.

$t = \min\{\frac{3}{12}, \frac{4}{6}, \text{-}\} = \frac{1}{4}$. x_4 leaves.

Update $x_3^* = t = \frac{1}{4}$, $x_5^* = 4 - (6)(\frac{1}{4}) = \frac{5}{2}$, $x_6^* = 2 - (-4)(\frac{1}{4}) = 3$.

<u>Iteration 2:</u>

$B = \{3, 5, 6\}$, $x_B^* = \begin{bmatrix} 1/4 \\ 5/2 \\ 3 \end{bmatrix}$, $A_B = \begin{bmatrix} 12 & 0 & 0 \\ 6 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$.

Solve $A_B^T y = c_B = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ to get $y = \begin{bmatrix} 3/4 \\ -1 \\ 1 \end{bmatrix}$.

Compute $\bar{c}_1 = c_1 - A_1^T y = -1 - \begin{bmatrix} -8 & -2 & 3 \end{bmatrix} y = 0 \leq 0$.

Compute $\bar{c}_2 = c_2 - A_2^T y = 1 - \begin{bmatrix} -3 & -1 & 1 \end{bmatrix} y = 5/4 > 0$.

x_2 enters.

Solve $A_B d = A_2$ to get $d = \begin{bmatrix} -1/4 \\ 1/2 \\ 0 \end{bmatrix}$.

$t = \min\{\text{-}, \frac{5/2}{1/2}, \text{-}\} = 5$. x_5 leaves.

Update $x_2^* = t = 5$, $x_3^* = \frac{1}{4} - (-\frac{1}{4})(5) = \frac{3}{2}$, $x_6^* = 3 - (0)(5) = 3$.

Iteration 3:

$B = \{2, 3, 6\}$, $x_B^* = \begin{bmatrix} 5 \\ 3/2 \\ 3 \end{bmatrix}$, $A_B = \begin{bmatrix} -3 & 12 & 0 \\ -1 & 6 & 0 \\ 1 & -4 & 1 \end{bmatrix}$.

Solve $A_B^T y = c_B = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ to get $y = \begin{bmatrix} -1/2 \\ 3/2 \\ 1 \end{bmatrix}$.

Compute $\bar{c}_1 = c_1 - A_1^T y = -1 - [\ -8 \quad -2 \quad 3\ ] y = -5 \leq 0$.

Compute $\bar{c}_4 = c_4 - A_4^T y = -1 - [\ 1 \quad 0 \quad 0\ ] y = -1/2 \leq 0$.

Compute $\bar{c}_5 = c_5 - A_5^T y = -1 - [\ 0 \quad 1 \quad 0\ ] y = -5/2 \leq 0$.

An optimal solution is $x^* = [0, 5, \frac{3}{2}, 0, 0, 3]^T$.

Note: $y^* = [-\frac{1}{2}, \frac{3}{2}, 1]^T$ is optimal for the dual LP.

Proof of optimality

$(-\frac{1}{2}) \times$ (eqn. 1) $+ (\frac{3}{2}) \times$ (eqn. 2) $+ (1) \times$ (eqn. 3):

$$4x_1 + x_2 - x_3 - \tfrac{1}{2}x_4 + \tfrac{3}{2}x_5 + x_6 = \tfrac{13}{2}$$

$\hat{x}$ is feasible $\implies$ $-x_1 + x_2 - x_3 - x_4 - x_5 + x_6$

$$\leq 4x_1 + x_2 - x_3 - \tfrac{1}{2}x_4 + \tfrac{3}{2}x_5 + x_6 = \tfrac{13}{2}$$

So optimal value $\leq \frac{13}{2}$.

$x^* = [0, 5, \frac{3}{2}, 0, 0, 3]^T$ is feasible and has value $\frac{13}{2}$

$\implies$ it is optimal.