CO350 Linear Programming Chapter 8: Degeneracy and Finite Termination

24th June 2005

Recap

The perturbation method

$$(P) \qquad \begin{array}{lrcl} \max & c^T x & & \\ \text{s.t.} & Ax & = & b \\ & x & \geq & 0 \end{array}$$

Assumption: B is a feasible basis with $A_B = I$.

Perturb the right hand side to $b' = b + [\varepsilon, \varepsilon^2, \ldots, \varepsilon^m]^T$ to get

$$(P') \qquad \begin{array}{lrcl} \max & c^T x & & \\ \text{s.t.} & Ax & = & b' \\ & x & \geq & 0 \end{array}$$

We showed that B is also a feasible basis of (P').

Tableaux for (P') and (P) differ in right hand side only $\Longrightarrow$ choices of leaving variables are affected.

Lemma 8.2

If ε is positive and sufficiently small, then

$$\alpha_0 + \alpha_1\varepsilon + \alpha_2\varepsilon^2 + \cdots + \alpha_m\varepsilon^m < \beta_0 + \beta_1\varepsilon + \beta_2\varepsilon^2 + \cdots + \beta_m\varepsilon^m$$

$$\iff \qquad (\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_m) \overset{L}{<} (\beta_0, \beta_1, \beta_2, \ldots, \beta_m)$$

Example (cycling example on pg 107)

Initial tableau:

$$\begin{array}{rrrrrrrl} z - 2x_1 & - 3x_2 & + x_3 & + 12x_4 & & & = & 0 \\ - 2x_1 & - 9x_2 & + x_3 & + 9x_4 & + x_5 & & = & 0 \\ \frac{1}{3}x_1 & + x_2 & - \frac{1}{3}x_3 & - 2x_4 & & + x_6 & = & 0 \end{array}$$

Tableau for perturbed problem:

$$\begin{array}{rrrrrrrl} z - 2x_1 & - 3x_2 & + x_3 & + 12x_4 & & & = & 0 \\ - 2x_1 & - 9x_2 & + x_3 & + 9x_4 & + x_5 & & = & \varepsilon \\ \frac{1}{3}x_1 & + x_2 & - \frac{1}{3}x_3 & - 2x_4 & & + x_6 & = & \varepsilon^2 \end{array}$$

$\bar{c}_2$ is largest positive reduced cost, so x_2 enters.
$\min\{-, \varepsilon^2/1\} = \varepsilon^2$, so x_6 leaves. Pivot on $(6,2)$:

$$\begin{array}{rrrrrrrl} z - x_1 & & & + 6x_4 & & + 3x_6 & = & 3\varepsilon^2 \\ x_1 & & - 2x_3 & - 9x_4 & + x_5 & + 9x_6 & = & \varepsilon + 9\varepsilon^2 \\ \frac{1}{3}x_1 & + x_2 & - \frac{1}{3}x_3 & - 2x_4 & & + x_6 & = & \varepsilon^2 \end{array}$$

$\bar{c}_1$ is only positive reduced costs, so x_1 enters.
$\min\{(\varepsilon + 9\varepsilon^2)/1, \varepsilon^2/\frac{1}{3}\} = 3\varepsilon^2$, so x_2 leaves. Pivot on $(2,1)$:

$$\begin{array}{rrrrrrrl} z & + 3x_2 & - x_3 & & & + 6x_6 & = & 6\varepsilon^2 \\ & - 3x_2 & - x_3 & - 3x_4 & + x_5 & + 6x_6 & = & \varepsilon + 6\varepsilon^2 \\ x_1 & + 3x_2 & - x_3 & - 6x_4 & & + 3x_6 & = & 3\varepsilon^2 \end{array}$$

The perturbed problem is unbounded.

Same pivots on original problem gives same conclusion.

Theorem 8.3 (pg 111)

(a) (P') is nondegenerate.

(b) B is a feasible basis of (P')
$\implies$ B is a feasible basis of (P).

(c) B is an optimal basis of (P')
$\implies$ B is an optimal basis of (P).

(d) x_k can enter and x_r can leave in tableau for (P') corresponding to B
$\implies$ same for tableau for (P) corresponding to B.

(e) Tableau for (P') corresponding to B detects unboundedness
$\implies$ same for tableau for (P) corresponding to B.

Proof of Theorem 8.3

(a) (P') is nondegenerate.

Proof: (Contradiction)

Suppose (P') has degenerate basis B.

Let x^* be basic solution of (P') determined by B.

So x^* solves $A_B x^*_B = b'$ and $x^*_N = 0$.

I.e., $x^*_B = A_B^{-1} b'$ and $x^*_N = 0$.

B degenerate $\Longrightarrow$ x^*_B has a zero component (say the h-th component is zero).

$$\begin{aligned} 0 &= h\text{-th component of } x^*_B \\ &= (h\text{-th row of } A_B^{-1})b' \\ &= [\alpha_1, \alpha_2, \ldots, \alpha_m] \left(b + \begin{bmatrix} \varepsilon \\ \varepsilon^2 \\ \vdots \\ \varepsilon^m \end{bmatrix} \right) \\ &= \alpha_0 + \alpha_1 \varepsilon + \alpha_2 \varepsilon^2 + \cdots + \alpha_m \varepsilon^m \end{aligned}$$

So by Lemma 8.2, $\alpha_i = 0$ for $i = 0, 1, 2, \ldots, m$.
Hence $[\alpha_1, \alpha_2, \ldots, \alpha_m] = (h\text{-th row of } A_B^{-1})$ is a zero row.

This contradicts A_B^{-1} is nonsingular. ∎

Proof of Theorem 8.3

(b) B is a feasible basis of (P')
$\implies$ B is a feasible basis of (P).

Proof:

Let x^* be the basic solution of (P') determined by B.
Let $\hat{x}$ be the basic solution of (P) determined by B.

B feasible for $(P') \implies x_i^* \geq 0$ for all $i \in B$.

Part (a) $\implies$ B is nondegenerate $\implies x_i^* > 0$ for all $i \in B$

$\hat{x}_i = (h\text{-th row of } A_B^{-1})b$

$$x_i^* = (h\text{-th row of } A_B^{-1})\left(b + \begin{bmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{bmatrix}\right)$$

$$= \hat{x}_i + \alpha_1\varepsilon + \cdots + \alpha_m\varepsilon^m$$

For all $i \in B$, $\quad x_i^* > 0 \implies (\hat{x}_i, \alpha_1, \ldots, \alpha_m) \overset{L}{>} (0, 0, \ldots, 0)$

$$\implies \hat{x}_i \geq 0$$

Thus $\hat{x}_i \geq 0$ for all $i \in B \implies B$ feasible for (P). ■

Proof of Theorem 8.3

(c) B is an optimal basis of (P')
$\implies$ B is an optimal basis of (P).

Proof:

Let (T') be the tableau for (P') corresponding to B.
Let (T) be the tableau for (P) corresponding to B.

B optimal for (P')
$\implies$ B feasible for (P') and all $\bar{c}_j$ in (T') are ≤ 0.

Part (b) $\implies$ B feasible for (P).

$\bar{c}_j$ are the same in both (T') and (T)
$\implies$ all $\bar{c}_j$ in (T) are ≤ 0.

B feasible for (P) and all $\bar{c}_j$ in (T) are ≤ 0
$\implies$ B optimal for (P). ∎

Proof of Theorem 8.3

(d) x_k can enter and x_r can leave in tableau for (P') corresponding to B

$\Longrightarrow$ same for tableau for (P) corresponding to B.

Proof:

Let (T') be the tableau for (P') corresponding to B.

Let (T) be the tableau for (P) corresponding to B.

x_k can enter in (T')

$\Longrightarrow$ $\bar{c}_k > 0$ in (T')

$\Longrightarrow$ $\bar{c}_k > 0$ in (T)

$\Longrightarrow$ x_k can enter in (T)

x_r can leave in (T')

$\Longrightarrow$ $\bar{a}_{rk} > 0$ in (T') and $\bar{b}_r / \bar{a}_{rk} =$ min. ratio

$\Longrightarrow$ $\bar{a}_{rk} > 0$ in (T') and new basis is feasible for (P')

$\Longrightarrow$ $\bar{a}_{rk} > 0$ in (T) and new basis is feasible for (P)

$\Longrightarrow$ x_r can leave in (T) ■

Proof of Theorem 8.3

(e) Tableau for (P') corresponding to B detects unboundedness

$\implies$ same for tableau for (P) corresponding to B.

Proof:

Let (T') be the tableau for (P') corresponding to B.

Let (T) be the tableau for (P) corresponding to B.

(T') detects unboundedness

$\implies$ $\bar{c}_k > 0$ in (T')

and $\bar{a}_{ik}$ in (T') are ≤ 0 for all $i \in B$

$\implies$ $\bar{c}_k > 0$ in (T) (from part (d))

and $\bar{a}_{ik}$ in (T) are ≤ 0 for all $i \in B$

$\implies$ (T) detects unboundedness

We have proved

Theorem 8.3 (pg 111)

(a) (P') is nondegenerate.

(b) B is a feasible basis of (P')
$\implies$ B is a feasible basis of (P).

(c) B is an optimal basis of (P')
$\implies$ B is an optimal basis of (P).

(d) x_k can enter and x_r can leave in tableau for (P') corresponding to B
$\implies$ same for tableau for (P) corresponding to B.

(e) Tableau for (P') corresponding to B detects unboundedness
$\implies$ same for tableau for (P) corresponding to B.

Corollary 8.3 (pg 112)

The simplex method applied to the perturbed problem (P') starting from a feasible basis B with $A_B = I$ will terminate after a finite number of iterations. Moreover,
B' optimal for (P') $\implies$ B' optimal for (P), and
(P') unbounded $\implies$ (P) unbounded.

The Lexicographical Simplex Method

It is an implementation of the simplex method on the perturbed problem (P').

We established that

> the difference between (P) and (P') is the choice of leaving variables.

Moreover

> all pivots on (P') can be performed on (P).

Conclusion:

> Simplex method on (P') is the same as simplex method on (P) with a special choice rule for leaving variables.

This special rule is called the lexicographical rule.

The resulting simplex method is called the lexicographical simplex method.

Lexicographical rule

R.h.s. of x_i-row (T') is

$$\bar{b}_i' = \bar{b}_i + \beta_{i1}\varepsilon + \beta_{i2}\varepsilon^2 + \cdots + \beta_{im}\varepsilon^m$$

where $[\beta_{i1}, \beta_{i2}, \ldots, \beta_{im}]$ is the h-th row of the matrix A_B^{-1}

and i is the h-th index in the basis B.

In choosing leaving variable, we pick r such that

$$\bar{a}_{rk} > 0 \quad \text{and} \quad \frac{\bar{b}_r'}{\bar{a}_{rk}} = \min\left\{\frac{\bar{b}_i'}{\bar{a}_{ik}} : \bar{a}_{ik} > 0\right\}$$

I.e., we pick the minimum of

$$\frac{\bar{b}_i + \beta_{i1}\varepsilon + \beta_{i2}\varepsilon^2 + \cdots + \beta_{im}\varepsilon^m}{\bar{a}_{ik}} \quad \text{over} \quad \{i \in B : \bar{a}_{ik} > 0\}$$

I.e., we pick the lexicographical minimum of

$$\frac{(\bar{b}_i, \beta_{i1}, \beta_{i2}, \ldots, \beta_{im})}{\bar{a}_{ik}} \quad \text{over} \quad \{i \in B : \bar{a}_{ik} > 0\}$$

All we need are $\bar{A}$, $\bar{b}$ and A_B^{-1}.

Note: This always give a unique choice:
Otherwise the next tableau is degenerate
(but we know that (P') is nondegenerate).

A_B^{-1} appears in the tableau!

We assumed initial basis B' has $A_{B'} = I$.

In the tableau corresponding to current basis B,

the x_i-rows are $$A_B^{-1}Ax = A_B^{-1}b$$

i.e. $$\bar{A} = A_B^{-1}A \quad \text{and} \quad \bar{b} = A_B^{-1}b$$

Magically, $$\bar{A}_{B'} = A_B^{-1}A_{B'} = A_B^{-1}$$

i.e., A_B^{-1} appears in the tableau corresponding to B as columns indexed by B'.

Example (Not in notes)

Solve using lexicographical simplex method.

Initial tableau:

$$\begin{array}{ccccccccccccc} z & - & x_1 & - & 2x_2 & & & & & & & = & 0 \\ & & 2x_1 & + & 4x_2 & + & 6x_3 & + & x_4 & & & = & 6 \\ & & x_1 & + & 3x_2 & + & 3x_3 & & & + & x_5 & = & 3 \end{array}$$

$\bar{c}_2$ is the largest positive reduced cost, so x_2 enters.
$\min\left\{\frac{(6,1,0)}{4}, \frac{(3,0,1)}{3}\right\} = \left(1, 0, \frac{1}{3}\right)$, so x_5 leaves.

Pivot on $(5,2)$:

$$\begin{array}{rrrrrrrrrrrrr} z & - & \frac{1}{3}x_1 & & & + & 2x_3 & & & + & \frac{2}{3}x_5 & = & 2 \\ & & \frac{2}{3}x_1 & & & + & 2x_3 & + & x_4 & - & \frac{4}{3}x_5 & = & 2 \\ & & \frac{1}{3}x_1 & + & x_2 & + & x_3 & & & + & \frac{1}{3}x_5 & = & 1 \end{array}$$

$\bar{c}_1$ is the only positive reduced cost, so x_1 enters.

$\min\left\{\frac{(2,1,-4/3)}{2/3}, \frac{(1,0,1/3)}{1/3}\right\} = (3,0,1)$, so x_2 leaves.

Pivot on $(2,1)$:

$$\begin{array}{rrrrrrrrrrr} z & & & + & x_2 & + & 3x_3 & & & + & x_5 & = & 3 \\ & & & - & 2x_2 & & & + & x_4 & - & 2x_5 & = & 0 \\ & & x_1 & + & 3x_2 & + & 3x_3 & & & + & x_5 & = & 3 \end{array}$$

This tableau is optimal.