CO350 Linear Programming Chapter 8: Degeneracy and Finite Termination

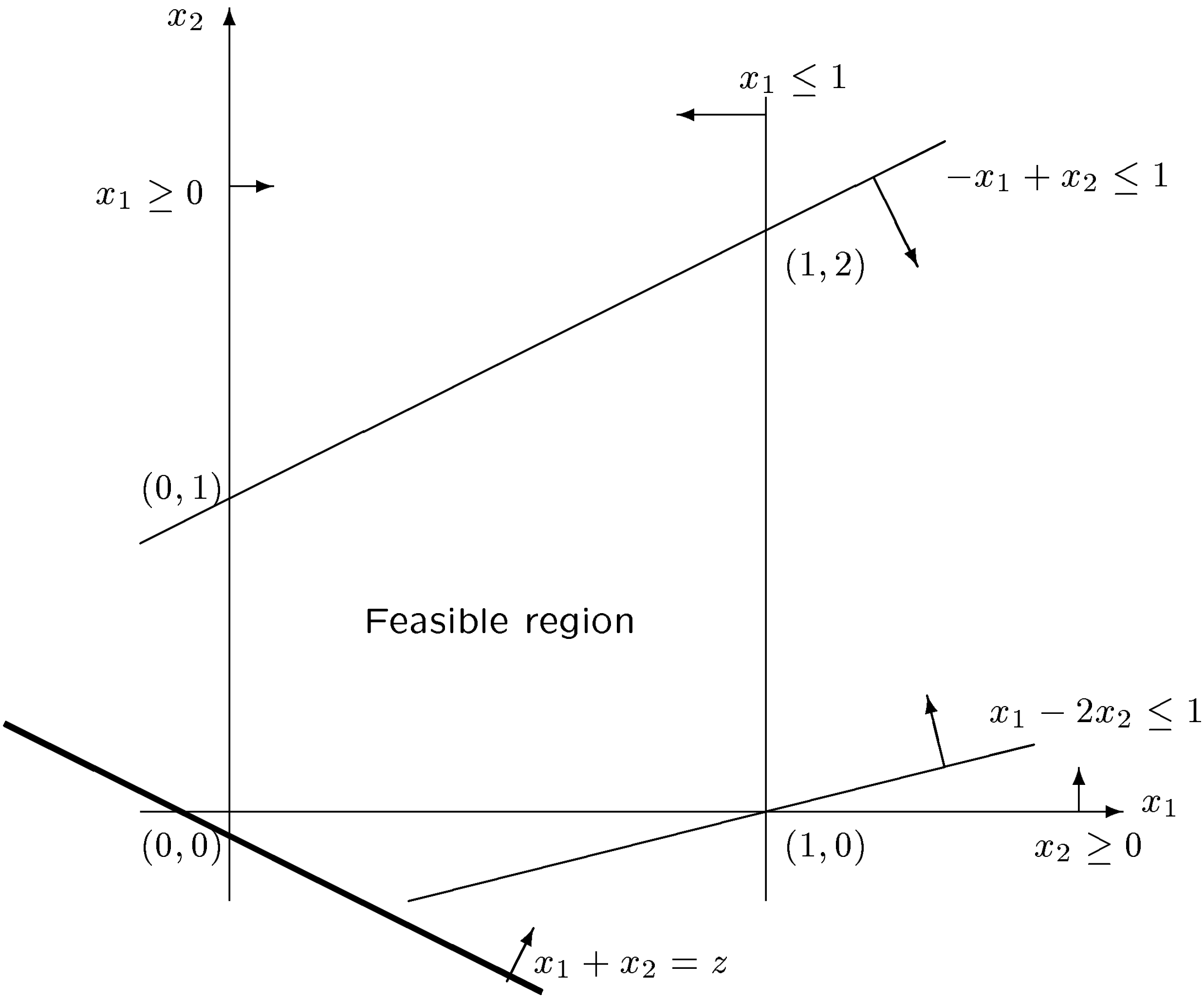
22th June 2005

Recap

On Monday, we established

- In the absence of degeneracy, the simplex method will terminate after a finite number of iterations.
- The following observations
  1. Iteration degenerate $\Longrightarrow$ old tableau degenerate. The converse is not true.
  2. Iteration degenerate $\Longrightarrow$ new tableau degenerate. The converse is not true.
  3. More than one choice of leaving variable $\Longrightarrow$ new basis degenerate.
     The converse is not true.
  4. Iteration is degenerate $\Longleftrightarrow$ basic solution remains the same.

Example of degeneracy (pg 105)



LP problem:

$$\begin{array}{lrcrcrcl} \text{max} & z & = & x_1 & + & x_2 & & \\ \text{s.t.} & & & -x_1 & + & x_2 & \le & 1 \\ & & & x_1 & - & 2x_2 & \le & 1 \\ & & & x_1 & & & \le & 1 \\ & & & x_1 & , & x_2 & \ge & 0 \end{array}$$

Geometry of degeneracy (bottom of pg 106)

From the example:

degeneracy in 2-dimension is represented by having more than two lines intersecting at an extreme point.

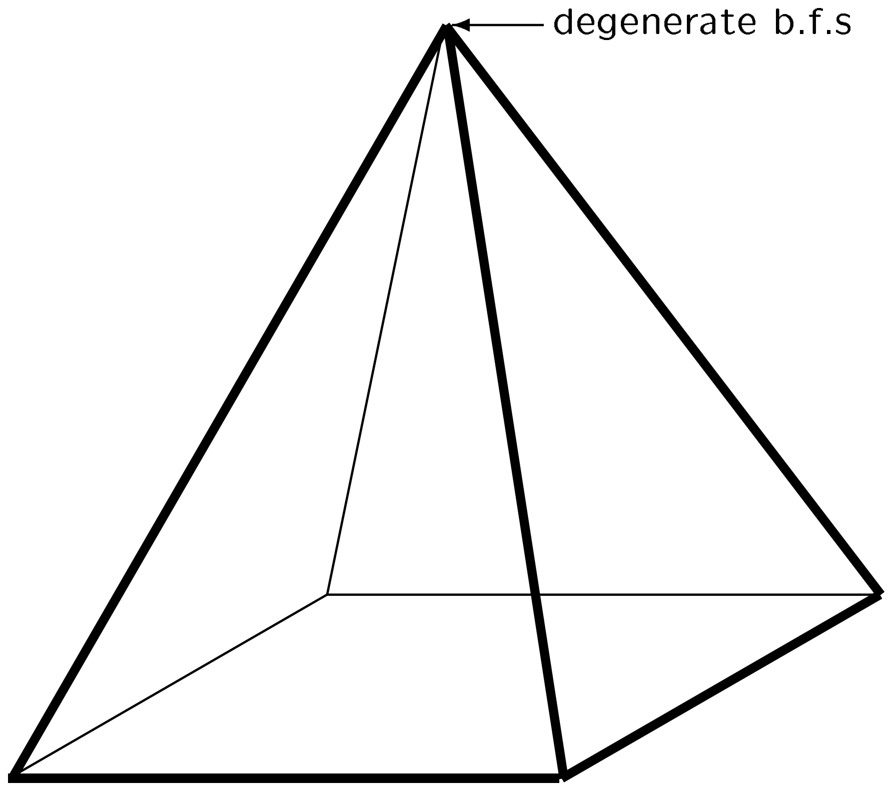
$\Longrightarrow$ there is a redundant constraint.

In general:

degeneracy in d-dimension is represented by having more than d hyperplanes intersecting at an extreme point.

However, this does not necessarily mean that there is a redundant constraint.

Example



Cycling

In some extreme cases, degeneracy may stall the simplex method **indefinitely**: We may visit the same sequence of bases over and over again.

This phenomenon is called cycling.

Example of cycling (pg 107)

Using largest coefficient rule for entering variable.

Initial tableau:

$$\begin{array}{lrrrrrrr}
z & -2x_1 & -3x_2 & +x_3 & +12x_4 & & & = 0 \\
 & -2x_1 & -9x_2 & +x_3 & +9x_4 & +x_5 & & = 0 \\
 & \frac{1}{3}x_1 & +x_2 & -\frac{1}{3}x_3 & -2x_4 & & +x_6 & = 0
\end{array}$$

Pivot on $(6,2)$:

$$\begin{array}{lrrrrrrr}
z & -x_1 & & & +6x_4 & & +3x_6 & = 0 \\
 & x_1 & & -2x_3 & -9x_4 & +x_5 & +9x_6 & = 0 \\
 & \frac{1}{3}x_1 & +x_2 & -\frac{1}{3}x_3 & -2x_4 & & +x_6 & = 0
\end{array}$$

Pivot on $(5,1)$:

$$\begin{array}{lrrrrrrr}
z & & & -2x_3 & -3x_4 & +x_5 & +12x_6 & = 0 \\
 & x_1 & & -2x_3 & -9x_4 & +x_5 & +9x_6 & = 0 \\
 & & x_2 & +\frac{1}{3}x_3 & +x_4 & -\frac{1}{3}x_5 & -2x_6 & = 0
\end{array}$$

Example of cycling (cont'd)

Pivot on $(2,4)$:

$$\begin{array}{rrrrrrrl} z & & +3x_2 & -x_3 & & & +6x_6 & = 0 \\ & x_1 & +9x_2 & +x_3 & & -2x_5 & -9x_6 & = 0 \\ & & x_2 & +\frac{1}{3}x_3 & +x_4 & -\frac{1}{3}x_5 & -2x_6 & = 0 \end{array}$$

Pivot on $(1,3)$:

$$\begin{array}{rrrrrrrl} z & +x_1 & +12x_2 & & & -2x_5 & -3x_6 & = 0 \\ & x_1 & +9x_2 & +x_3 & & -2x_5 & -9x_6 & = 0 \\ & -\frac{1}{3}x_1 & -2x_2 & & +x_4 & +\frac{1}{3}x_5 & +x_6 & = 0 \end{array}$$

Pivot on $(4,6)$:

$$\begin{array}{rrrrrrrl} z & & +6x_2 & & +3x_4 & -x_5 & & = 0 \\ & -2x_1 & -9x_2 & +x_3 & +9x_4 & +x_5 & & = 0 \\ & -\frac{1}{3}x_1 & -2x_2 & & +x_4 & +\frac{1}{3}x_5 & +x_6 & = 0 \end{array}$$

Pivot on $(3,5)$:

$$\begin{array}{rrrrrrrl} z & -2x_1 & -3x_2 & +x_3 & +12x_4 & & & = 0 \\ & -2x_1 & -9x_2 & +x_3 & +9x_4 & +x_5 & & = 0 \\ & \frac{1}{3}x_1 & +x_2 & -\frac{1}{3}x_3 & -2x_4 & & +x_6 & = 0 \end{array}$$

This is the same as the initial tableau!

The Perturbation Method (§8.3)

In this chapter, we consider the LP problem

$$(P) \quad \begin{array}{lrcl} \max & c^T x & & \\ \text{s.t.} & Ax & = & b \\ & x & \geq & 0 \end{array}$$

and assume that we are given a feasible basis B.

Motivation

Degeneracy occurs because too many hyperplanes intersect at an extreme point.

Solution: nudge each hyperplane slightly so that such intersection disappear.

More precisely, we change b to

$$b' = \begin{bmatrix} b_1 + \varepsilon \\ b_2 + \varepsilon^2 \\ \vdots \\ b_m + \varepsilon^m \end{bmatrix}$$

where ε is a very small but positive real number, and consider the LP problem

$$(P') \quad \begin{array}{lrcl} \max & c^T x & & \\ \text{s.t.} & Ax & = & b' \\ & x & \geq & 0 \end{array}$$

We need to ensure that

1. the new LP problem (P') is feasible, and
2. the new LP problem (P') is nondegenerate.

1. (P') is feasible

If $A_B = I$, then the b.f.s. of $\{Ax = b',\ x \geq 0\}$ determined by B is

$$x_B = b', \qquad x_N = 0,$$

which is feasible.

If $A_B \neq I$, then we first change the equality constraints of (P) to

$$(A_B^{-1}A)x = A_B^{-1}b$$

Feasible region remains unchanged.

The new equality constraints $\widehat{A}x = \widehat{b}$ satisfies $\widehat{A}_B = I$.

Then we change $\widehat{b}$ to $\widehat{b}'$ by adding $\varepsilon, \varepsilon^2, \ldots, \varepsilon^m$.

In both cases, (P') is feasible.

Before showing that (P') is nondegenerate, consider

Question: How can we apply the simplex method without knowing the value ε?

Answer: ε only appears on the r.h.s. of each tableau

$\Longrightarrow$ it only affects the choice of leaving variable

$\Longrightarrow$ we only need to compare numbers of the form

$$d_0 + d_1\varepsilon + d_2\varepsilon^2 + \cdots + d_m\varepsilon^m$$

Good news: Such numbers are comparable if ε is small enough.

Lemma 8.2 (pg 110). Consider the numbers

$$\alpha = \alpha_0 + \alpha_1\varepsilon + \alpha_2\varepsilon^2 + \cdots + \alpha_m\varepsilon^m$$

and

$$\beta = \beta_0 + \beta_1\varepsilon + \beta_2\varepsilon^2 + \cdots + \beta_m\varepsilon^m$$

For all ε sufficiently small and positive,

1. $\alpha = \beta$ if and only if $\alpha_i = \beta_i$ for $0 \le i \le m$.
2. $\alpha > \beta$ if and only if $\alpha_k > \beta_k$ where k is the least integer i for which $\alpha_i \neq \beta_i$.
3. $\alpha < \beta$ if and only if $\alpha_k < \beta_k$ where k is the least integer i for which $\alpha_i \neq \beta_i$.

Example (Not in notes)

Arrange the following numbers in ascending order, assuming that ε is very small and positive.

$$\begin{array}{lcrrrr}
p_1 & = & & 3\varepsilon & - & 13\varepsilon^2 \\
p_2 & = & & 3\varepsilon & + & 10\varepsilon^2 \\
p_3 & = & & -\ 100\varepsilon & + & 5\varepsilon^2 \\
p_4 & = & 2 & & + & 17\varepsilon^2 \\
p_5 & = & 2 & -\ 7\varepsilon & &
\end{array}$$

Comparing constants: $p_1, p_2, p_3 < p_4, p_5$.

Comparing ε terms: $p_5 < p_4$ and $p_3 < p_1, p_2$.

Comparing ε^2 terms: $p_1 < p_2$.

Answer: $p_3 < p_1 < p_2 < p_5 < p_4$

Lexicographic ordering:

Suppose we represent p_1 by $(0, 3, -13)$

and p_2 by $(0, 3, 10)$.

Then $p_1 < p_2$ corresponds to the fact that $(0, 3, -13)$ is lexicographically less than $(0, 3, 10)$.

We write $(0, 3, -13) \overset{L}{<} (0, 3, 10)$

This is also precisely how we order words in dictionaries:

cap cat go golf to

We need to ensure that

1. the new LP problem (P') is feasible, and
2. the new LP problem (P') is nondegenerate.

2. (P') is nondegenerate

Theorem 8.3 (pg 111)

(a) (P') is nondegenerate.

(b) B is a feasible basis of (P')
$\Longrightarrow$ B is a feasible basis of (P).

(c) B is an optimal basis of (P')
$\Longrightarrow$ B is an optimal basis of (P).

(d) x_k can enter and x_r can leave in tableau for (P') corresponding to B
$\Longrightarrow$ same for tableau for (P) corresponding to B.

(e) Tableau for (P') corresponding to B detects unboundedness
$\Longrightarrow$ same for tableau for (P) corresponding to B.

Corollary 8.3 (pg 112)

The simplex method applied to the perturbed problem (P') starting from a feasible basis B with $A_B = I$ will terminate after a finite number of iterations. Moreover,
B' optimal for (P') $\Longrightarrow$ B' optimal for (P), and
(P') unbounded $\Longrightarrow$ (P) unbounded.