Review of Part III: Solving LP problems

We learned how to solve LP problems using

- the simplex method and
- the two-phase method.

We also studied

- basic feasible solutions and
- extreme points.

By now, you should be able to

- give the definitions of basis, basic solutions, b.f.s., convex sets, extreme points and tableau,
- prove the relation between b.f.s. and extreme points,
- explain why we focus on b.f.s.,
- determine optimal solution or unboundedness,
- prove optimality and unboundedness algebraically,
- construct auxiliary problems and determine feasibility of LP problems,
- prove infeasibility algebraically.

CO350 Linear Programming Chapter 8: Degeneracy and Finite Termination

20th June 2005

Motivation

In the next 2 weeks, we aim at proving

Fundamental Theorem of Linear Programming.

For any LP problem, exactly one of the following is true:

1. it has an optimal solution;
2. it is infeasible;
3. it is unbounded.

The main tool: two-phase method.

Upon successful completion of the two-phase method, we have one of the above three conclusions [Moreover, we have algebraic proofs of each of them.]

We only need to ensure the successful completion of simplex method.

We shall see that

- degeneracy may prevent the successful completion of simplex method;
- perturbing the LP can remove degeneracy;
- using appropriate choice rules for leaving variables can ensure successful completion.

Degeneracy

A simple proof that the simplex method will stop:

- There are a finite number of feasible bases (at most n choose m).
- Every feasible basis determines a b.f.s. with an objective value.
- IF we strictly increase the objective value at each step, then we never use a feasible basis twice.
- Conclusion: after at most n choose m steps, simplex method will stop.

> Note: the proof requires that we strictly increase the objective value.

Change in objective value after each pivot $= \bar{v} + \bar{c}_k t$, where $t =$ min. ratio $= \bar{b}_r / \bar{a}_{rk}$.

Increase in obj. value is strict

$$\iff \quad \bar{c}_k t > 0$$

$$\iff \quad t = \bar{b}_r / \bar{a}_{rk} > 0 \qquad \text{(because } \bar{c}_k > 0\text{)}$$

$$\iff \quad \bar{b}_r > 0 \qquad \text{(because } \bar{a}_{rk} > 0\text{)}$$

So, $\bar{b}_r = 0$ in any pivot will invalidate our proof.

> $\bar{b}_r = 0$ means the basic solution is degenerate.

Definitions

Bad case: $\bar{b}_r = 0$ (or equivalently, $t = 0$).

(Defn) **Degenerate iteration (or step, or pivot)**
An iteration (or step, or pivot) where the objective value is not strictly increased;
i.e. $\bar{b}_r = 0$, or equivalently, $t = 0$.

(Defn) **Degenerate basic solution**
(defined in Chapter 5, pg 62)
A basic solution that has less than m non-zeros.
i.e., $x_i = 0$ for some $i \in B$.

(Defn) **Degenerate basis**
A basis that determines a degenerate basic solution.

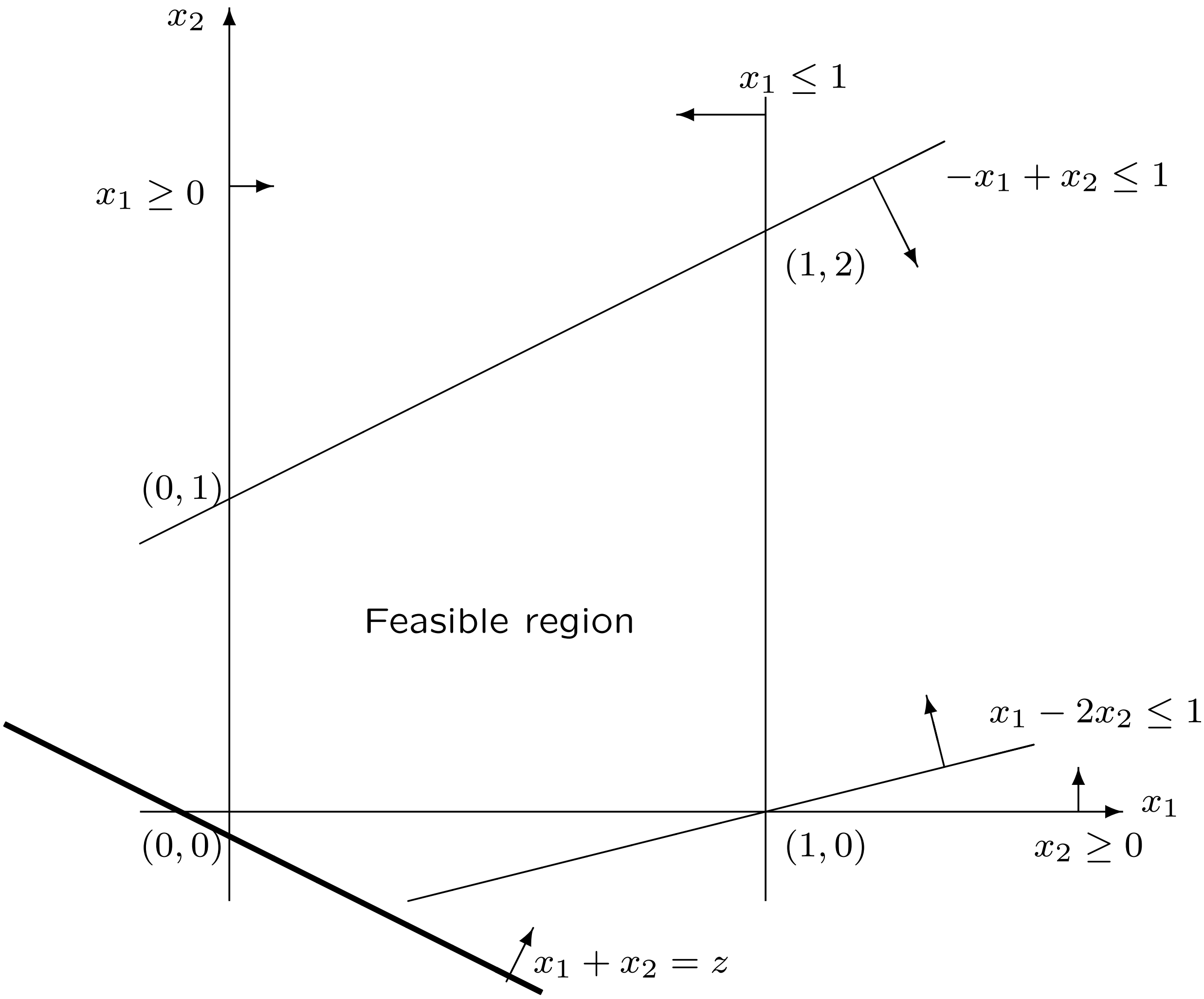
(Defn) **Degenerate tableau**
A tableau corresponding to a degenerate basis.
i.e., $\bar{b}_i = 0$ for some $i \in B$.

An iteration/basic solution/basis/tableau is nondegenerate if it is not degenerate.

(Defn) **Degenerate linear programming problem**
An LP problem that does not have any degenerate basis.

[Note: This is actually a property of its feasible region.]

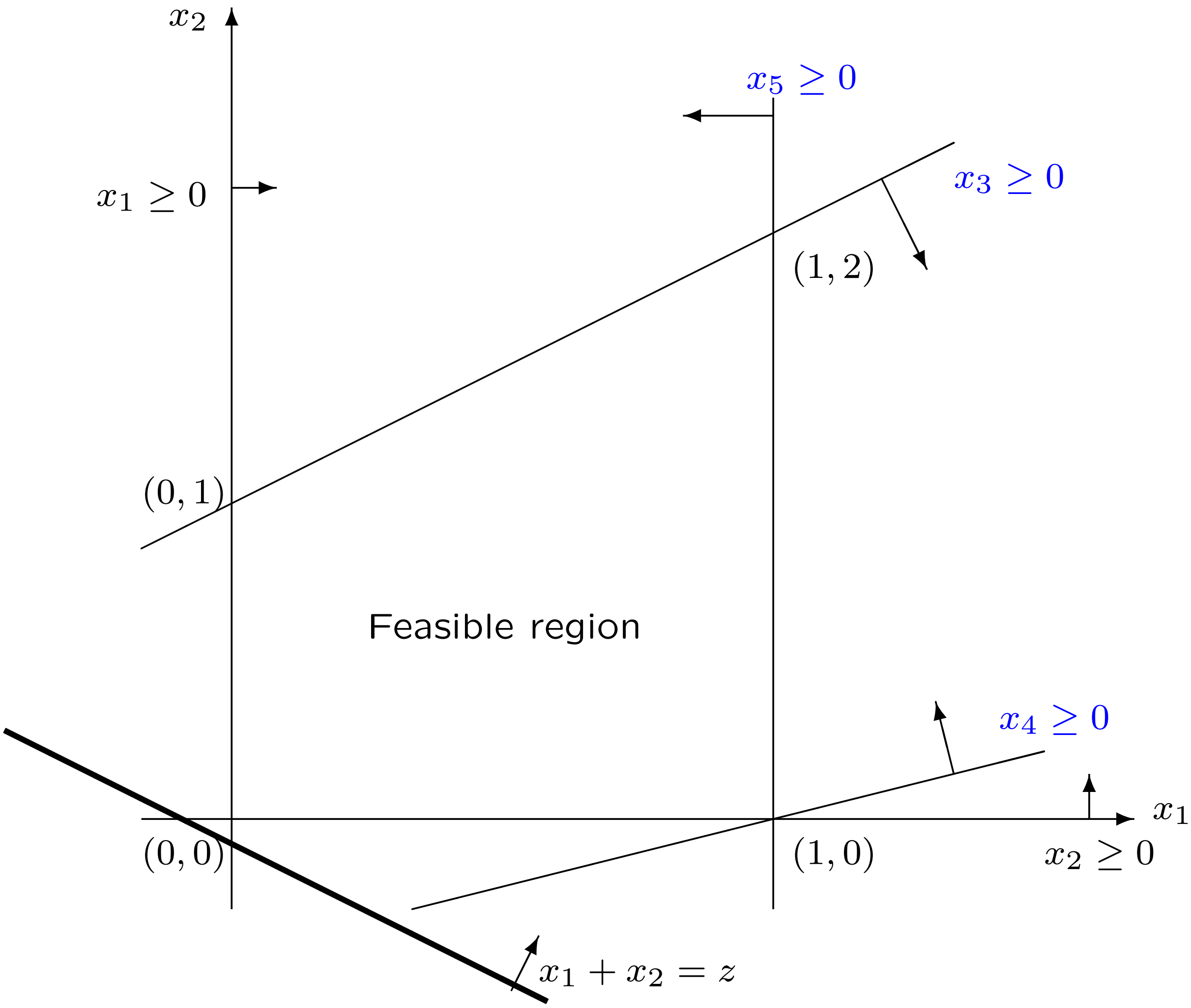
Example of degeneracy (pg 105)



LP problem:

$$
\begin{array}{llrcrcl}
\text{max} & z = & x_1 & + & x_2 & & \\
\text{s.t.} & & -x_1 & + & x_2 & \leq & 1 \\
& & x_1 & - & 2x_2 & \leq & 1 \\
& & x_1 & & & \leq & 1 \\
& & x_1 & , & x_2 & \geq & 0
\end{array}
$$

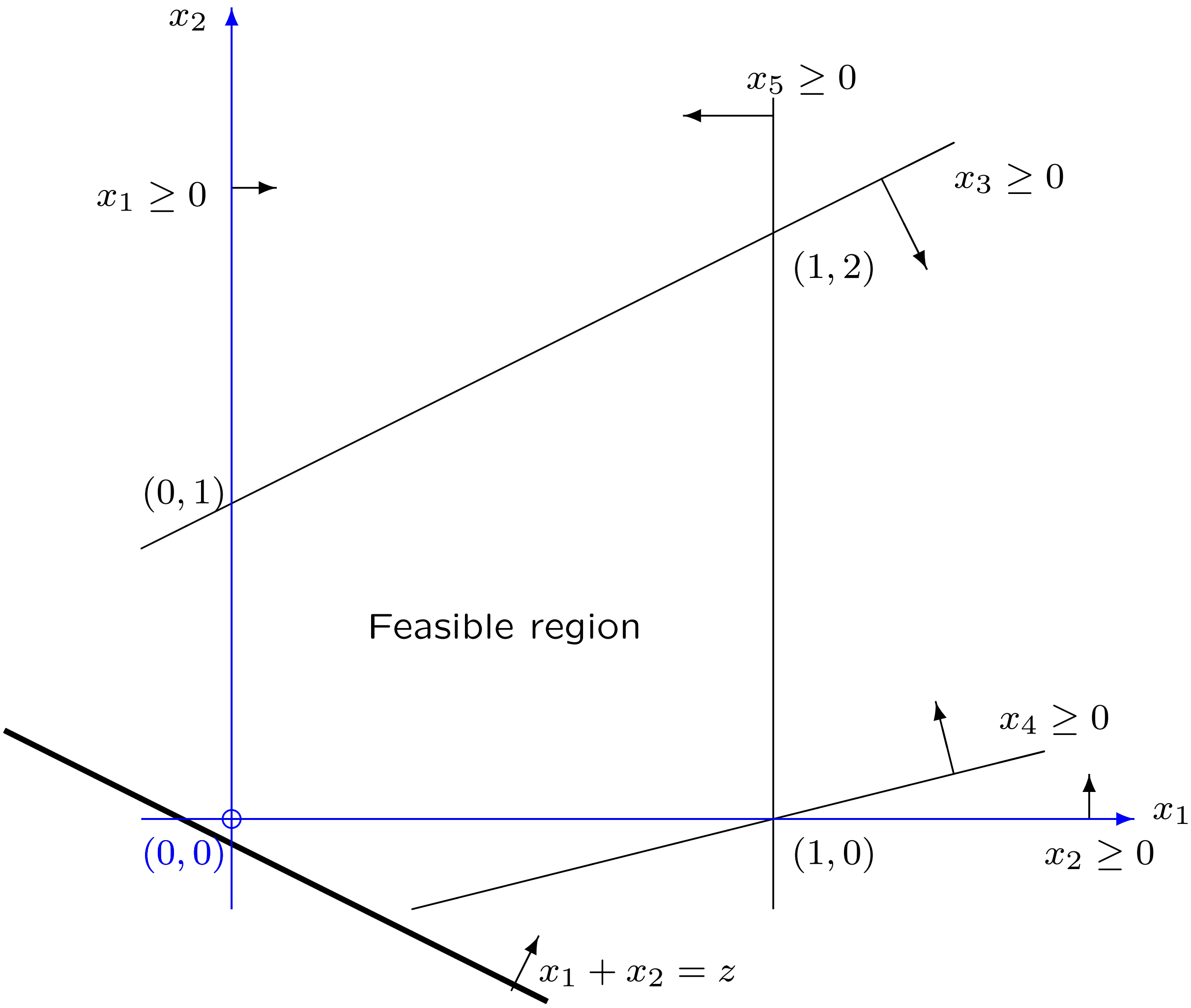
Example of degeneracy (pg 105)



Adding slack variables:

$$
\begin{array}{llrcrcrcrcrcl}
\max & z = & x_1 & + & x_2 & & & & & & & & \\
\text{s.t.} & & -x_1 & + & x_2 & + & x_3 & & & & & = & 1 \\
& & x_1 & - & 2x_2 & & & + & x_4 & & & = & 1 \\
& & x_1 & & & & & & & + & x_5 & = & 1 \\
& & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq & 0
\end{array}
$$

Example of degeneracy (pg 105)



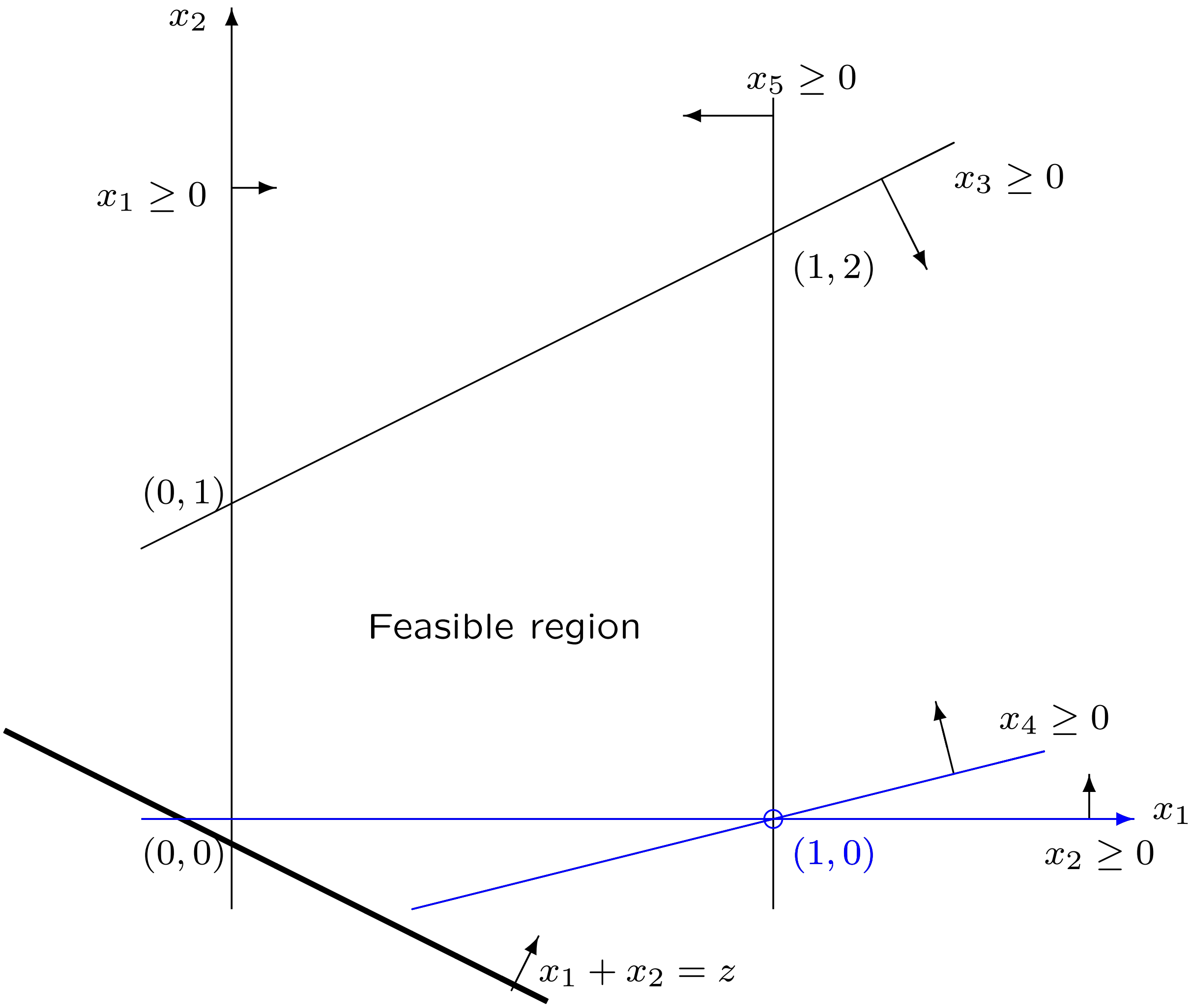
Initial tableau:

$$
\begin{array}{rcrcrcrcrcrcl}
z & - & x_1 & - & x_2 & & & & & & & = & 0 \\
 & & -x_1 & + & x_2 & + & x_3 & & & & & = & 1 \\
 & & x_1 & - & 2x_2 & & & + & x_4 & & & = & 1 \\
 & & x_1 & & & & & & & + & x_5 & = & 1
\end{array}
$$

$\bar{c}_1 = 1 > 0$, so x_1 enters.

$t = \min\{-, 1/1, 1/1\} = 1$, so x_4 leaves. (x_5 may also leave).

Example of degeneracy (pg 105)



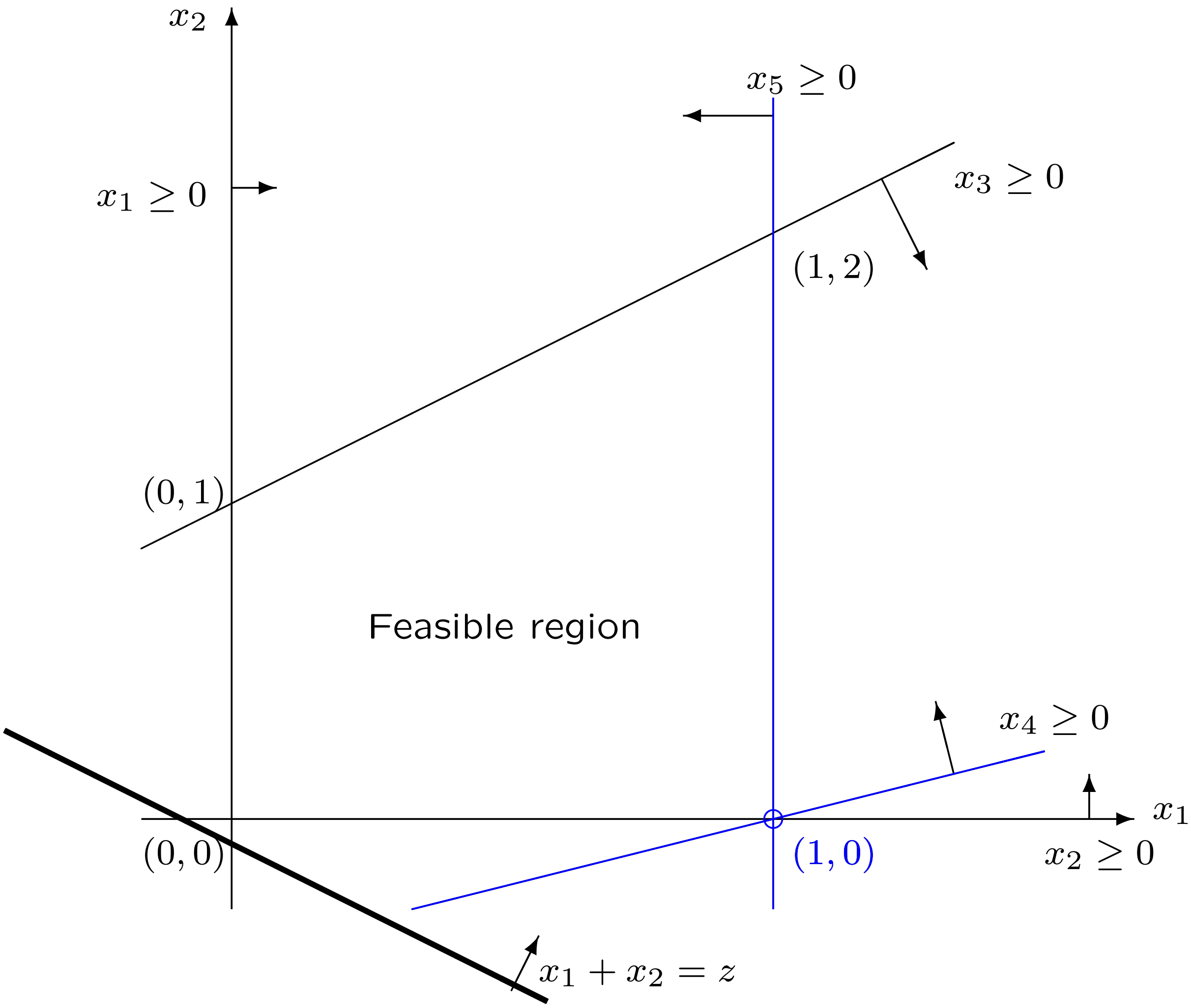
Pivot on (4,1):

$$
\begin{array}{rrrrrrrrrrr}
z & & - & 3x_2 & & & + & x_4 & & & = & 1 \\
 & & & -x_2 & + & x_3 & & & & & = & 2 \\
 & x_1 & - & 2x_2 & & & + & x_4 & & & = & 1 \\
 & & & 2x_2 & & & - & x_4 & + & x_5 & = & 0
\end{array}
$$

$\bar{c}_2 = 3 > 0$, so x_2 enters.

$t = \min\{-, -, 0/2\} = 0$, so x_5 leaves.

Example of degeneracy (pg 105)



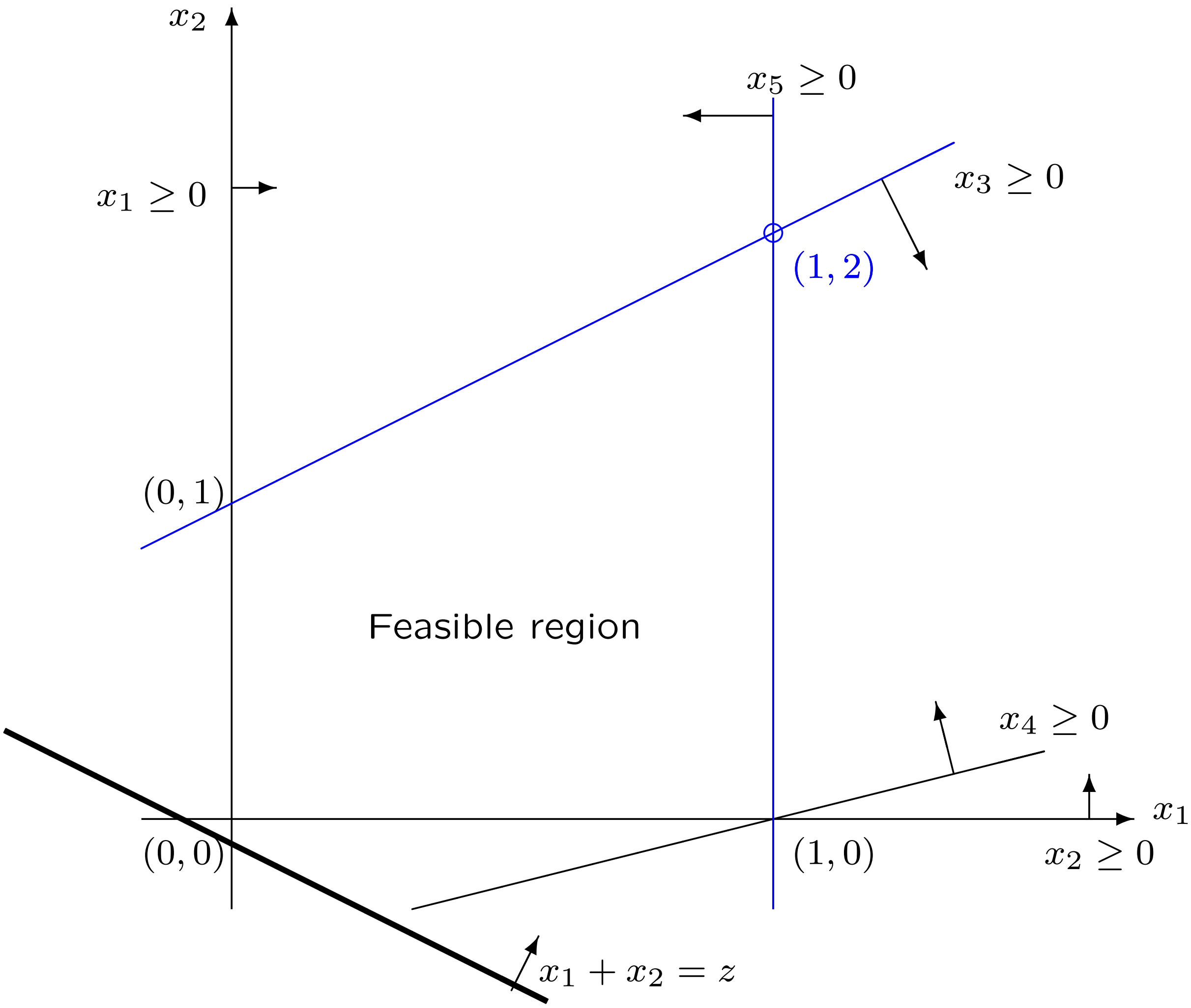
Pivot on (5,2):

$$
\begin{array}{rcrcrcrcl}
z & & & - & \frac{1}{2}x_4 & + & \frac{3}{2}x_5 & = & 1 \\
 & & x_3 & + & \frac{1}{2}x_4 & + & \frac{1}{2}x_5 & = & 2 \\
x_1 & & & & & + & x_5 & = & 1 \\
 & x_2 & & - & \frac{1}{2}x_4 & + & \frac{1}{2}x_5 & = & 0
\end{array}
$$

$\bar{c}_4 = \frac{1}{2} > 0$, so x_4 enters.

$t = \min\{2/\frac{1}{2}, -, -\} = 4$, so x_3 leaves.

Example of degeneracy (pg 105)



Pivot on (3,4):

$$\begin{array}{rrrrrrrrrrr} z & & & + & x_3 & & & + & 2x_5 & = & 3 \\ & & & & 2x_3 & + & x_4 & + & x_5 & = & 4 \\ & x_1 & & & & & & + & x_5 & = & 1 \\ & & x_2 & + & x_3 & & & + & x_5 & = & 2 \end{array}$$

This tableau is optimal.

Example of degeneracy (pg 105)

Initial tableau:

$$\begin{aligned}
z - x_1 - x_2 &= 0\\
-x_1 + x_2 + x_3 &= 1\\
x_1 - 2x_2 + x_4 &= 1\\
x_1 + x_5 &= 1
\end{aligned}$$

Iteration 1:

$$\begin{aligned}
z - 3x_2 + x_4 &= 1\\
-x_2 + x_3 &= 2\\
x_1 - 2x_2 + x_4 &= 1\\
2x_2 - x_4 + x_5 &= 0
\end{aligned}$$

Iteration 2:

$$\begin{aligned}
z - \tfrac{1}{2}x_4 + \tfrac{3}{2}x_5 &= 1\\
x_3 + \tfrac{1}{2}x_4 + \tfrac{1}{2}x_5 &= 2\\
x_1 + x_5 &= 1\\
x_2 - \tfrac{1}{2}x_4 + \tfrac{1}{2}x_5 &= 0
\end{aligned}$$

Iteration 3:

$$\begin{aligned}
z + x_3 + 2x_5 &= 3\\
2x_3 + x_4 + x_5 &= 4\\
x_1 + x_5 &= 1\\
x_2 + x_3 + x_5 &= 2
\end{aligned}$$

Observations:

1. Iteration degenerate $\Longrightarrow$ old tableau degenerate.

   Proof: Iteration degenerate $\Longrightarrow$ $\bar{b}_r = 0$
   $\Longrightarrow$ x_r is basic in old basis, and $x_r = \bar{b}_r = 0$. ■

   Converse is not true (e.g., iteration 3).

2. Iteration degenerate $\Longrightarrow$ new tableau degenerate.

   Proof: Iteration degenerate $\Longrightarrow$ $t = 0$
   $\Longrightarrow$ x_k is basic in new basis, and $x_k = t = 0$. ■

   Converse is not true (e.g., iteration 1).

3. More than one choice of leaving variable $\Longrightarrow$ new basis degenerate.

   Proof: Suppose $\bar{b}_{r'}/\bar{a}_{r'k} = \bar{b}_r/\bar{a}_{rk} = t$.
   Then new $x_{r'} =$ old $x_{r'} - \bar{a}_{r'k}t = \bar{b}_{r'} - \bar{a}_{r'k}(\bar{b}_{r'}/\bar{a}_{r'k}) = 0$
   $\Longrightarrow$ $x_{r'}$ basic in new basis, and $x_{r'} = 0$. ■

   Converse is not true (e.g., iteration 2).

4. Iteration is degenerate $\Longleftrightarrow$ basic solution remains the same.

   Proof: Iteration degenerate $\Longrightarrow$ $t = 0$
   $\Longrightarrow$ basic solution unchanged.

   Basic solution unchanged $\Longrightarrow$ obj. value unchanged
   $\Longrightarrow$ iteration degenerate. ■

Theorem 8.1 (pg 104) If an LP problem is nondegenerate, then the simplex method applied to the problem starting from a feasible basis will terminate after a finite number of iterations.

Proof (similar to before)

- There are at most n choose m feasible bases.
- Nondegenerate LP

  $\Longrightarrow$ no degenerate basis (by definition)

  $\Longrightarrow$ no degenerate iteration (by observation 1 (or 2))

  $\Longrightarrow$ objective value increases strictly at each step (by definition)

- Conclusion: after at most n choose m steps, simplex method will stop.