CO350 Linear Programming Chapter 7: The Two-Phase Method

17th June 2005

Recap

$$(P) \quad \begin{array}{llrcrcrcrcl} \text{max} & (z=) & x_1 & & & & & & & & \\ \text{s.t.} & & 3x_1 & + & 5x_2 & + & 2x_3 & - & x_4 & = & 7 \\ & & 2x_1 & + & 5x_2 & + & 3x_3 & + & x_4 & = & 3 \\ & & x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0 \end{array}$$

So far this week, we learned

- to construct the auxiliary problem (A),
- to detect infeasibility by solving (A),
- to obtain a feasible basis of (P) from optimal tableau for (A),
- two observations that may simplify the construction and solution of (A).

An Exceptional Case (§7.4)

It may happen that

(A) has optimal value 0, and yet

the optimal tableau has basic artificial variables.

BUT we want a feasible basis for (P)!

I.e., **no artificial variables should be basic**.

Good news:

Optimal value is $0 \implies$ all artificial variables $= 0$.

So, every basic artificial variable has value 0.

(This, by definition, means that the bfs is degenerate.)

To "force" the artificial variable out of basis, we pivot!

Example (Pg 96)

Suppose that only x_5 is artificial.

$$\begin{array}{ccccccccccccc} w & & & & & & + & 2x_4 & & & = & 0 \\ & & x_2 & + & 4x_3 & + & 3x_4 & & & & = & 2 \\ & & & - & 2x_3 & - & 2x_4 & + & x_5 & & = & 0 \\ & x_1 & & + & 3x_3 & + & 2x_4 & & & & = & 1 \end{array}$$

We want x_5 to leave. We choose either x_3 or x_4 to enter.

BUT this violates the rules of the simplex method.

- Neither $\bar{c}_3$ nor $\bar{c}_4$ are positive.
- Neither $\bar{a}_{53}$ nor $\bar{a}_{54}$ are positive.

HOWEVER, $\bar{b}_5 = 0 \implies$ r.h.s. values will not change.
i.e., obj. value remains 0, and tableau remains feasible.

Pivoting on $(5, 4)$ (and dropping the artificial x_5) gives

$$\begin{array}{ccccccccc} w & & & - & 2x_3 & & & = & 0 \\ & & x_2 & + & x_3 & & & = & 2 \\ & & & & x_3 & + & x_4 & = & 0 \\ & x_1 & & + & x_3 & & & = & 1 \end{array}$$

Notice the tableau becomes non-optimal.
But $B = \{1, 2, 4\}$ contains no artificial variable
$\implies$ it is feasible for the original LP.
So we can start Phase 2 with $B = \{1, 2, 4\}$.

Example (Pg 96)

Suppose that only x_5 is artificial.

$$\begin{array}{ccccccccccc} w & & & & & & & & & = & 0 \\ & & x_2 & & & + & x_4 & & & = & 2 \\ & & & & & & & & x_5 & = & 0 \\ & x_1 & & + & 2x_3 & + & x_4 & & & = & 1 \end{array}$$

We want x_5 to leave.

But there is nothing to pivot on. $(\bar{a}_{53} = \bar{a}_{54} = 0)$

Observe:

- $x_5 = 0$ is obtained from the equality constraints of (A) via elementary row operations.
- Performing same row operations on original $Ax = b$ gives a zero row!
- So, there is a redundant constraint to begin with.

Remedy:

- Drop $x_5 = 0$ since it is redundant.

Two-Phase Method: The Algorithm (Pg 97)

1. Construct the auxiliary problem (A).
   [Try to use slack variables as much as possible.]

2. (Phase 1) Solve (A) using simplex method, dropping artificial variables as soon as they become nonbasic.
   [Stop immediately once the objective value $\bar{v} = 0$.]

3. (Infeasibility Test) If optimal value of (A) is < 0. Stop and conclude original LP (P) is infeasible.

4. (Construct Feasible Basis) While there exists a basic artificial variable x_i,
   (a) if the x_i-row is $x_i = 0$, remove it from tableau and drop i from basis.
   (b) if $\bar{a}_{ik} \neq 0$ for some k, pivot on (i, k) and drop x_i.

5. (Update Objective Row) Get z-row by eliminating all basic variables from $z - c^T x = 0$, and replace w-row with z-row.

6. (Phase 2) Begin with tableau from Step 5, solve (P) using simplex method to either get an optimal solution or conclude that (P) is unbounded.

Feasibility Theorems and Simplex Method (§7.6)

$$(P) \qquad \begin{array}{lrcl} \max & c^T x & & \\ \text{s.t.} & Ax & = & b \\ & x & \geq & 0 \end{array}$$

Suppose (P) is infeasible.

In practice, we can use Phase 1 to detect the infeasibility.

In theory, Farkas' Lemma says there is a y that proves the infeasibility.

Farkas' Lemma. Let A be an $m \times n$ matrix.
The system

$$Ax = b, \;\; x \geq 0$$

has no solution $\iff$ there exists $y = [y_1, y_2, \ldots, y_m]^T$ such that

$$A^T y \geq 0 \text{ and } b^T y < 0.$$

We shall see that

$\boxed{\text{The vector } y \text{ can be obtained at the end of Phase 1.}}$

First we assume that $b \geq 0$.

We add artificial variables to get the auxiliary problem

$$(A) \qquad \begin{array}{llrcrcll} \max & (w =) & -\displaystyle\sum_{i=1}^{m} x_{n+i} & & & & & \\ \text{s.t.} & \displaystyle\sum_{j=1}^{n} a_{ij}x_j & + & x_{n+i} & = & b & (i = 1, 2, \ldots, m) \\ & & & x_j & \geq & 0 & (j = 1, 2, \ldots, m+n) \end{array}$$

In matrix notation, (A) is

$$\max \quad d^T x, \quad \text{s.t.} \quad Dx = b, \; x \geq 0,$$

where $d = [0, 0, \ldots, 0, -1, -1, \ldots, -1]^T$ and $D = [A \mid I]$.

Suppose the final optimal tableau for (A) is

$$(T) \qquad \begin{array}{ccccl} w & - & \sum_{j \in N} \bar{c}_j x_j & = & \bar{v} \\ x_i & + & \sum_{j \in N} \bar{a}_{ij} x_j & = & \bar{b}_i \quad (i \in B) \end{array}$$

Since (P) is infeasible, it must be that $\bar{v} < 0$.

By definition of a tableau, (T) is obtained from

$$(L) \qquad \begin{array}{rcl} w - d^T x & = & 0 \\ Dx & = & b \end{array}$$

via elementary row operations.

So equations in (T) are linear combos of equations in (L).

$$(T) \qquad \begin{array}{rcrcl} w & - & \sum_{j\in N} \bar{c}_j x_j & = & \bar{v} \quad (< 0) \\ x_i & + & \sum_{j\in N} \bar{a}_{ij} x_j & = & \bar{b}_i \quad (i \in B) \end{array}$$

$$(L) \qquad \begin{array}{rcl} w - d^T x & = & 0 \\ Dx & = & b \end{array}$$

Equations in (T) are linear combos of equations in (L)

So there exists y_0, $y = [y_1, \ldots, y_m]^T$ such that

$$\begin{array}{rrcl} & w \; - & \sum_{j\in N} \bar{c}_j x_j & = \bar{v} \\ \equiv & y_0 w \; - & y_0 d^T x + y^T D x & = y^T b \end{array}$$

Comparing coefficient of w gives $y_0 = 1$.

Comparing coefficient of x_i $(i \in B)$ gives

$$-d_i + D_i^T y = 0 \qquad (i \in B)$$

So y solves $D_B^T y = d_B$.

Comparing coefficient of x_j $(j = 1, 2, \ldots, n)$ gives

$$\begin{array}{lrcll} & -d_j + D_j^T y & = & -\bar{c}_j & (j = 1, 2, \ldots, n) \\ \text{i.e.,} & A_j^T y & = & -\bar{c}_j \geq 0 & (j = 1, 2, \ldots, n) \\ \text{i.e.,} & A^T y & \geq & 0 & \end{array}$$

Comparing right hand side gives

$$y^T b = \bar{v}$$

$$\text{i.e.,} \quad b^T y = \bar{v} < 0$$

In summary, if we solve for y in $D_B^T y = d_B$, where

$$d = [0, 0, \dots, 0, -1, -1, \dots, -1]^T \quad \text{and} \quad D = [A \mid I]$$

and B is an optimal basis for (A),

then y satisfies

$$A^T y \geq 0 \quad \text{and} \quad b^T y < 0;$$

i.e., y is the vector in Farkas' Lemma.

Example (Not in notes)

Recall the example of infeasible LP problem from §7.2.

$$(P) \quad \begin{array}{ll} \max & (z =) \; x_1 \\ \text{s.t.} & 3x_1 + 5x_2 + 2x_3 - x_4 = 7 \\ & 2x_1 + 5x_2 + 3x_3 + x_4 = 3 \\ & x_1 \;,\; x_2 \;,\; x_3 \;,\; x_4 \geq 0 \end{array}$$

The auxiliary problem is

$$(A) \quad \begin{array}{ll} \max & (w =) \; - x_5 - x_6 \\ \text{s.t.} & 3x_1 + 5x_2 + 2x_3 - x_4 + x_5 \qquad = 7 \\ & 2x_1 + 5x_2 + 3x_3 + x_4 \qquad + x_6 = 3 \\ & x_1 \;,\; x_2 \;,\; x_3 \;,\; x_4 \;,\; x_5 \;,\; x_6 \geq 0 \end{array}$$

The optimal basis is $B = \{1, 5\}$.

We solve for y in $D_B^T y = d_B$, where

$$d = [0,0,0,0,-1,-1]^T \quad \text{and} \quad D = \begin{bmatrix} 3 & 5 & 2 & 1 & 1 & 0 \\ 2 & 5 & 3 & 1 & 0 & 1 \end{bmatrix},$$

and $B = \{1, 5\}$

Solving

$$\begin{array}{rcrcr} 3y_1 & + & 2y_2 & = & 0 \\ y_1 & & & = & -1 \end{array}$$

gives $y_1 = -1$, $y_2 = 3/2$.

This gives a proof of infeasibility as follows:

For feasible x, $(-1) \times$ (eqn.1) $+ (3/2) \times$ (eqn.2) gives

$$\frac{5}{2}x_2 + \frac{5}{2}x_3 + \frac{5}{2}x_4 = -\frac{5}{2}$$

which contradicts $x_2, x_3, x_4 \geq 0$.

Suppose b has one or more negative components.
Let $J = \{i : b_i < 0\}$.

We should multiply equation i of $Ax = b$ by -1 for all $i \in J$ before constructing (A).

This gives the new system $A'x = b'$, and
we can find y satisfying $(A')^T y \geq 0$ and $(b')^T y < 0$.

Let $\hat{y}_i = \begin{cases} -y_i & \text{if } i \in J, \\ y_i & \text{if } i \notin J. \end{cases}$
Then $\hat{y}$ satisfies $A^T \hat{y} \geq 0$ and $b^T \hat{y} < 0$.

Here is why: (slightly different from notes)

Let a_i be i-th row of A and a'_i be i-th row of A'.

Therefore $[a'_i \mid b'_i] = \begin{cases} [-a_i \mid -b_i] & \text{if } i \in J, \\ [a_i \mid b_i] & \text{if } i \notin J. \end{cases}$

Suppose y satisfies $(A')^T y \geq 0$ and $(b')^T y < 0$.

$$\begin{aligned} A^T \hat{y} &= \sum_{i=1}^{m} \hat{y}_i a_i^T = \sum_{i \in J} (-y_i)(-a'_i)^T + \sum_{i \notin J} y_i (a'_i)^T \\ &= \sum_{i=1}^{m} y_i (a'_i)^T = (A')^T y \geq 0 \end{aligned}$$

$$b^T \hat{y} = \sum_{i=1}^{m} \hat{y}_i b'_i = \sum_{i \in J} (-y_i)(-b_i) + \sum_{i \notin J} y_i b_i = (b')^T y < 0$$