CO350 Linear Programming Chapter 7: The Two-Phase Method

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Recap

To construct auxiliary problem (A) for

$$(P) \qquad \begin{array}{lrcl} \text{max} & c^T x & & \\ \text{s.t.} & Ax & = & b \\ & x & \geq & 0 \end{array}$$

we

- make sure that $b \geq 0$,
- introduce one artificial variable u_i for each constraint, and
- change the objective to $-\sum_{i=1}^{m} u_i$.

After solving (A),

- if (A) has optimal value < 0, (P) is infeasible;
- if (A) has optimal value $= 0$, we can get a feasible basis for (P) from the optimal tableau for (A).

Infeasibility (§7.2)

Example of infeasibility

Given the LP problem

$$
(P)\quad
\begin{array}{llrcrcrcrcl}
\text{max} & (z =) & x_1 & & & & & & & & \\
\text{s.t.} & & 3x_1 & + & 5x_2 & + & 2x_3 & - & x_4 & = & 7 \\
 & & 2x_1 & + & 5x_2 & + & 3x_3 & + & x_4 & = & 3 \\
 & & x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0
\end{array}
$$

The auxiliary problem is

$$
(A)\quad
\begin{array}{llrcrcrcrcrcrcl}
\text{max} & (w =) & & & & & & & & - & x_5 & - & x_6 & & \\
\text{s.t.} & & 3x_1 & + & 5x_2 & + & 2x_3 & - & x_4 & + & x_5 & & & = & 7 \\
 & & 2x_1 & + & 5x_2 & + & 3x_3 & + & x_4 & & & + & x_6 & = & 3 \\
 & & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & , & x_6 & \geq & 0
\end{array}
$$

Tableau corresponding to $B = \{5, 6\}$ is

$$
\begin{array}{crcrcrcrcrcrcr}
w & - & 5x_1 & - & 10x_2 & - & 5x_3 & & & & & & & = & -10 \\
 & & 3x_1 & + & 5x_2 & + & 2x_3 & - & x_4 & + & x_5 & & & = & 7 \\
 & & 2x_1 & + & 5x_2 & + & 3x_3 & + & x_4 & & & + & x_6 & = & 3
\end{array}
$$

Note: the w-row is obtained by subtracting x_5-row and x_6-row from $w = -x_5 - x_6$.

Example (cont'd)

$\bar{c}_1 = 5 > 0$, so x_1 enters. $t = \min\{7/3, 3/2\} = 3/2$, so x_6 leaves. Pivot on $(6,1)$ gives the tableau

$$\begin{array}{rrrrrrrrrrrrl}
w & & + & \frac{5}{2}x_2 & + & \frac{5}{2}x_3 & + & \frac{5}{2}x_4 & & & + & \frac{5}{2}x_6 & = -\frac{5}{2} \\
 & & - & \frac{5}{2}x_2 & - & \frac{5}{2}x_3 & - & \frac{5}{2}x_4 & + & x_5 & - & \frac{3}{2}x_6 & = \frac{5}{2} \\
 & x_1 & + & \frac{5}{2}x_2 & + & \frac{3}{2}x_3 & + & \frac{1}{2}x_4 & & & + & \frac{1}{2}x_6 & = \frac{3}{2}
\end{array}$$

This tableau is optimal.

The optimal value is $-5/2 < 0 \implies (P)$ is infeasible.

Observations (§7.3)

Two observations that may simplify Phase 1:

1. When an artificial variable becomes nonbasic, we can drop it from the problem.
2. When there is a variable that behaves like a slack variable for a constraint, we can do without an artificial variable for that constraint.

Example

$$(P) \quad \begin{array}{lrcrcrcl} \text{maximize} & 2x_1 & + & x_2 & + & 3x_3 & & \\ \text{subject to} & x_1 & + & x_2 & + & x_3 & \geq & 2 \\ & x_1 & - & x_2 & & & = & 1 \\ & x_1 & & & - & x_3 & \leq & 2 \\ & x_1 & , & x_2 & , & x_3 & \geq & 0 \end{array}$$

Convert to SEF:

$$(P') \quad \begin{array}{lrcrcrcrcrcl} \text{maximize} & 2x_1 & + & x_2 & + & 3x_3 & & & & & & \\ \text{subject to} & x_1 & + & x_2 & + & x_3 & - & x_4 & & & = & 2 \\ & x_1 & - & x_2 & & & & & & & = & 1 \\ & x_1 & & & - & x_3 & & & + & x_5 & = & 2 \\ & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq & 0 \end{array}$$

Auxiliary problem (A):

$$
\begin{array}{lccccccccccccccccc}
\max & & & & & & & & & & & - & x_6 & - & x_7 & & \\
\text{s.t.} & x_1 & + & x_2 & + & x_3 & - & x_4 & & & & + & x_6 & & & = & 2 \\
& x_1 & - & x_2 & & & & & & & & & & + & x_7 & = & 1 \\
& x_1 & & & - & x_3 & & & + & x_5 & & & & & & = & 2 \\
& x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & , & x_6 & , & x_7 & & \geq & 0
\end{array}
$$

Note that we do not need any artificial variable for the third constraint.

Initial tableau corresponding to basis $B = \{5, 6, 7\}$:

$$
\begin{array}{cccccccccccccccc}
w & - & 2x_1 & & & - & x_3 & + & x_4 & & & & & & = & -3 \\
& & x_1 & + & x_2 & + & x_3 & - & x_4 & & & + & x_6 & & = & 2 \\
& & x_1 & - & x_2 & & & & & & & & & + \; x_7 & = & 1 \\
& & x_1 & & & - & x_3 & & & + & x_5 & & & & = & 2
\end{array}
$$

$\bar{c}_1 = 2 > 0$, so x_1 enters. $t = \min\{2/1, 1/1, 2/1\} = 1$ so x_7 leaves. Pivot on $(7, 1)$ (and dropping the artificial x_7) gives

$$
\begin{array}{cccccccccccccc}
w & & & - & 2x_2 & - & x_3 & + & x_4 & & & & = & -1 \\
& & & & 2x_2 & + & x_3 & - & x_4 & & & + \; x_6 & = & 1 \\
& x_1 & & - & x_2 & & & & & & & & = & 1 \\
& & & & x_2 & - & x_3 & & & + & x_5 & & = & 1
\end{array}
$$

$\bar{c}_2 = 2 > 0$, so x_2 enters. $t = \min\{1/2, -, 1/1\} = 1/2$ so x_6 leaves. Pivot on $(6, 2)$ (and dropping the artificial x_6) gives

$$\begin{array}{lllllllll} w & & & & & & & = & 0 \\ & x_2 & + & \frac{1}{2}x_3 & - & \frac{1}{2}x_4 & & = & \frac{1}{2} \\ x_1 & & + & \frac{1}{2}x_3 & - & \frac{1}{2}x_4 & & = & \frac{3}{2} \\ & & - & \frac{3}{2}x_3 & + & \frac{1}{2}x_4 & + \ x_5 & = & \frac{1}{2} \end{array}$$

This tableau is optimal. Optimal value of (A) is 0.

The optimal basis $B = \{1, 2, 5\}$ is feasible for (P').

Tableau of (P') corresponding to $B = \{1, 2, 5\}$ is

$$\begin{array}{lllllllll} z & & - & \frac{3}{2}x_3 & - & \frac{3}{2}x_4 & & = & \frac{7}{2} \\ & x_2 & + & \frac{1}{2}x_3 & - & \frac{1}{2}x_4 & & = & \frac{1}{2} \\ x_1 & & + & \frac{1}{2}x_3 & - & \frac{1}{2}x_4 & & = & \frac{3}{2} \\ & & - & \frac{3}{2}x_3 & + & \frac{1}{2}x_4 & + \ x_5 & = & \frac{1}{2} \end{array}$$

From here on, you should be able to solve (P') easily with the simplex method.