CO350 Linear Programming Chapter 7: The Two-Phase Method

13th June 2005

Recap

In the past week and a half, we learned the simplex method and its relation with duality.

By now, you should know how to

- solve an LP problem given an initial feasible basis;
- give a proof of optimality/unboundedness from the final tableau;
- compute/read a dual optimal solution from an optimal tableau;
- relate dual optimal solution with shadow prices in the case of nondegeneracy.

Motivation

Consider the LP

$$(P) \qquad \begin{array}{rrcl} \max & c^T x & & \\ \text{s.t.} & Ax & = & b \\ & x & \geq & 0 \end{array}$$

We have assumed that a feasible basis is always given. But in practice, it is usually not easy to spot a feasible basis.

Duality theory says: optimal solutions to (P) and its dual are solutions to

$$Ax = b, \; x \geq 0$$
$$A^T y \geq c$$
$$c^T x - b^T y = 0$$

So, finding feasible solution is as hard as solving LP.

Two-phase method: an algorithm that solves (P) in two phases, where

- in Phase 1, we solve an auxiliary LP problem to either get a feasible basis or conclude that (P) is infeasible.
- in Phase 2, we solve (P) starting from the feasible basis found in Phase 1.

Remark: from Phase 1, we see that finding feasible basis is as easy as solving LP.

The Two-Phase Method (§7.1)

Artificial variables and auxiliary problem

Consider the LP

$$(P) \qquad \begin{array}{lrcl} \max & c^T x & & \\ \text{s.t.} & Ax & = & b \\ & x & \geq & 0 \end{array}$$

Assumption: $b \geq 0$. (This is without loss of generality.)

Suppose we relax the equality constraints to inequalities, and add slack variables $u_1, u_2, \ldots, u_m$.

$$\begin{array}{lrcccl} \max & c^T x & & & & \\ \text{s.t.} & Ax & + & u & = & b \\ & x & , & u & \geq & 0 \end{array}$$

The basis having $u_1, u_2, \ldots, u_m$ as basic variables is feasible; it determines the bfs $(x^*, u^*) = (0, b)$.

These "slack" variables are called artificial variables.

This new LP problem is NOT equivalent to (P).

BUT, if we can force all artificial variables to be zero, then the resulting solution gives a feasible solution to (P).

So, we change the objective function!

$$
(A) \qquad \begin{array}{lrcrcl}
\text{max} & & & -\sum_{i=1}^{m} u_i & & \\
\text{s.t.} & Ax & + & u & = & b \\
 & x & , & u & \geq & 0
\end{array}
$$

This is called an auxiliary problem.

Example

Given the LP problem

$$
(P) \qquad \begin{array}{llrcrcrcrcr}
\text{max} & (z =) & -x_1 & & & - & x_3 & + & 2x_4 & & \\
\text{s.t.} & & x_1 & + & 2x_2 & & & + & x_4 & = & 4 \\
 & & & & -x_2 & + & x_3 & - & x_4 & = & -1 \\
 & & x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0
\end{array}
$$

First we make sure the right hand side is nonnegative.

$$
(P) \qquad \begin{array}{llrcrcrcrcr}
\text{max} & (z =) & -x_1 & & & - & x_3 & + & 2x_4 & & \\
\text{s.t.} & & x_1 & + & 2x_2 & & & + & x_4 & = & 4 \\
 & & & & x_2 & - & x_3 & + & x_4 & = & 1 \\
 & & x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0
\end{array}
$$

Adding artificial variables u_1, u_2 gives the auxiliary problem

$$(A) \quad \begin{array}{lll} \text{max} & (w =) & \qquad\qquad\qquad\qquad - u_1 - u_2 \\ \text{s.t.} & & x_1 + 2x_2 \qquad + x_4 + u_1 \qquad = 4 \\ & & \qquad\quad x_2 - x_3 + x_4 \qquad + u_2 = 1 \\ & & x_1 , \; x_2 , \; x_3 , \; x_4 , \; u_1 , \; u_2 \geq 0 \end{array}$$

Any feasible solution of (A) has objective value ≤ 0.
$\Longrightarrow$ (A) has optimal value ≤ 0.

$[x_1^*, x_2^*, x_3^*, x_4^*]^T$ is feasible for (P),
$\Longrightarrow$ $[x_1^*, x_2^*, x_3^*, x_4^*, 0, 0]^T$ is feasible for (A).
$\Longrightarrow$ $[x_1^*, x_2^*, x_3^*, x_4^*, 0, 0]^T$ is optimal for (A) with value 0.

$[x_1^*, x_2^*, x_3^*, x_4^*, u_1^*, u_2^*]^T$ is optimal for (A) with value 0
$\Longrightarrow$ $u_1^* = u_2^* = 0$
$\Longrightarrow$ $[x_1^*, x_2^*, x_3^*, x_4^*]^T$ is feasible for (P).

So
(P) has a feasible solution $\iff$ (A) has optimal value 0.

$\boxed{\text{In general, the auxiliary problem is never unbounded; Its optimal value is } \leq 0.}$

Using the same argument as before, we can prove

Theorem 7.1 (Pg 91).

$$\begin{array}{ll} & \text{An LP problem } (P) \text{ has a feasible solution} \\ \Longleftrightarrow & \text{its auxiliary problem } (A) \text{ has an optimal value } 0. \end{array}$$

The two-phase method constructs and solves the auxiliary problem (A) in the first phase.

- if (A) has optimal value < 0, we conclude that (P) is infeasible.
- if (A) has optimal value $= 0$, we construct a feasible basis for (P) and solve it in the second phase.

Example (cont'd)

$$
\begin{array}{llrcrcrcrcl}
 & \text{max} & (z=) & -x_1 & & & - & x_3 & + & 2x_4 & \\
(P) & \text{s.t.} & & x_1 & + & 2x_2 & & & + & x_4 & = 4 \\
 & & & & & x_2 & - & x_3 & + & x_4 & = 1 \\
 & & & x_1 & , & x_2 & , & x_3 & , & x_4 & \geq 0
\end{array}
$$

$$
\begin{array}{llrcrcrcrcrcrl}
 & \text{max} & (w=) & & & & & & & & - & x_5 & - x_6 & \\
(A) & \text{s.t.} & & x_1 & + & 2x_2 & & & + & x_4 & + & x_5 & & = 4 \\
 & & & & & x_2 & - & x_3 & + & x_4 & & & + x_6 & = 1 \\
 & & & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & , x_6 & \geq 0
\end{array}
$$

[We let $x_5 = u_1$ and $x_6 = u_2$.]

We solve the auxiliary problem starting from the obvious feasible basis $B = \{5, 6\}$.

The corresponding tableau is

$$
\begin{array}{rcrcrcrcrcrcrcr}
w & - & x_1 & - & 3x_2 & + & x_3 & - & 2x_4 & & & & & = & -5 \\
 & & x_1 & + & 2x_2 & & & + & x_4 & + & x_5 & & & = & 4 \\
 & & & & x_2 & - & x_3 & + & x_4 & & & + & x_6 & = & 1
\end{array}
$$

Note: the w-row is obtained by subtracting x_5-row and x_6-row from $w = -x_5 - x_6$.

Example (cont'd)

$\bar{c}_1 = 1 > 0$, so x_1 enters. $t = \min\{4/1, -\} = 4$, so x_5 leaves. Pivot on $(5,1)$ gives the tableau

$$\begin{array}{rrrrrrrrrrrrrrr}
w & & & - & x_2 & + & x_3 & - & x_4 & + & x_5 & & & = & -1 \\
& & x_1 & + & 2x_2 & & & + & x_4 & + & x_5 & & & = & 4 \\
& & & & x_2 & - & x_3 & + & x_4 & & & + & x_6 & = & 1
\end{array}$$

$\bar{c}_2 = 1 > 0$, so x_2 enters. $t = \min\{4/2, 1/1\} = 1$, so x_6 leaves. Pivot on $(6,2)$ gives the tableau

$$\begin{array}{rrrrrrrrrrrrrr}
w & & & & & & & & x_5 & + & x_6 & = & 0 \\
& x_1 & & + & 2x_3 & - & x_4 & + & x_5 & - & 2x_6 & = & 2 \\
& & x_2 & - & x_3 & + & x_4 & & & + & x_6 & = & 1
\end{array}$$

This tableau is optimal, and $B = \{1, 2\}$ is an optimal basis.

$B = \{1, 2\}$ does not contain artificial variable
$\Longrightarrow$ $B = \{1, 2\}$ is a feasible basis for (P).

The tableau for (P) corresponding to $B = \{1, 2\}$ is

$$\begin{array}{rrrrrrrrr}
z & & & - & x_3 & - & x_4 & = & -2 \\
& x_1 & & + & 2x_3 & - & x_4 & = & 2 \\
& & x_2 & - & x_3 & + & x_4 & = & 1
\end{array}$$

Note: the z-row is obtained by eliminating the basic variables x_1 and x_2 from $z = -x_1 - x_3 + 2x_4$.

Example (cont'd)

$\bar{c}_3 = 1 > 0$, so x_3 enters. $t = \min\{2/2, -\} = 1$, so x_1 leaves. Pivot on $(1,3)$ gives the tableau

$$\begin{array}{rcrcrcrcrcr} z & + & \frac{1}{2}x_1 & & & & & - & \frac{3}{2}x_4 & = & -1 \\ & & \frac{1}{2}x_1 & & & + & x_3 & - & \frac{1}{2}x_4 & = & 1 \\ & & \frac{1}{2}x_1 & + & x_2 & & & + & \frac{1}{2}x_4 & = & 2 \end{array}$$

$\bar{c}_4 = \frac{3}{2} > 0$, so x_4 enters. $t = \min\{-, 2/1\} = 2$, so x_2 leaves. Pivot on $(2,4)$ gives the tableau

$$\begin{array}{rcrcrcrcrcr} z & + & 2x_1 & + & 3x_2 & & & & & = & 5 \\ & & x_1 & + & x_2 & + & x_3 & & & = & 3 \\ & & x_1 & + & 2x_2 & & & + & x_4 & = & 4 \end{array}$$

This tableau is optimal. The corresponding optimal solution is $x^* = [0, 0, 3, 4]^T$ with optimal value 5.