CO350 Linear Programming
Chapter 6: The Simplex Method

10th June 2005

Recap

On Wednesday, we learned four common choice rules for entering variables:

- Largest coefficient rule (Dantzig's rule);
- Smallest subscript rule;
- Largest improvement rule;
- Steepest edge rule.

We also learned that the simplex method solves the dual problem implicitly.

The Simplex Method and Duality (cont'd)

Suppose at the end of the simplex method, we have an optimal solution x^* determined by a basis B and the corresponding tableau

$$(T) \qquad \begin{array}{rcrcl} z & - & \sum_{j \in N} \bar{c}_j x_j & = & \bar{v} \\ x_i & + & \sum_{j \in N} \bar{a}_{ij} x_j & = & \bar{b}_i \quad (i \in B) \end{array}$$

Recall that

1. The z-row is
$$[z - c^T x = 0] + [\text{linear combination of } Ax = b]$$
I.e., there is some $\hat{y} = [\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_m]^T$ such that
$$z - \textstyle\sum_{j \in N} \bar{c}_j x_j = \bar{v}$$
is equivalent to
$$z - c^T x + y^T A x = y^T b$$

2. Comparing coefficients gives
$$c_i - A_i^T \hat{y} = 0 \quad (i \in B)$$
$$c_j - A_j^T \hat{y} = \bar{c}_j \quad (j \in N)$$
which shows that $\hat{y}$ is optimal for the dual problem.

Finding dual optimal solution

The above-mentioned dual optimal solution $\hat{y}$ satisfies

$$c_i - A_i^T y = 0 \quad (i \in B)$$

$$\text{i.e.} \qquad A_B^T y = c_B$$

[Note: c_B to denotes the column matrix $[c_i : i \in B]$.]

Since B is a basis, A_B is nonsingular, and so is A_B^T.

Thus the system $A_B^T y = c_B$ has the unique solution $\hat{y}$.

Example (Not in notes)

Recall the example LP.

$$
\begin{array}{lrcrcrcrcrcrcl}
\text{max.} & z & = & 5x_1 & + & 3x_2 & & & & & & & & \\
\text{s.t.} & & & 2x_1 & + & 3x_2 & + & x_3 & & & & & = & 15 \\
& & & 2x_1 & + & x_2 & & & + & x_4 & & & = & 9 \\
& & & x_1 & - & x_2 & & & & & + & x_5 & = & 3 \\
& & & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq & 0
\end{array}
$$

We have applied the simplex method to get the optimal basis $B = \{1, 2, 5\}$.

To find an optimal solution y^* for the dual problem, we solve the system $A_B^T y = c_B$;

i.e.,

$$
\begin{bmatrix} 2 & 3 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}^T \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_5 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 0 \end{bmatrix}
$$

or

$$
\begin{array}{rcrcrcl}
2y_1 & + & 2y_2 & + & y_3 & = & 5 \\
3y_1 & + & y_2 & - & y_3 & = & 3 \\
& & & & y_3 & = & 0
\end{array}
$$

The unique solution is $y^* = [1/4,\ 9/4,\ 0]^T$.

Finding dual optimal solution for SIF

For the special case of SIF, we can actually read the values of y^* from the optimal tableau!

Consider the LP problem

$$\begin{array}{lrcl} \text{max} & c^T x & & \\ \text{s.t.} & Ax & \leq & b \\ & x & \geq & 0 \end{array}$$

We add slack variables $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$ and apply the simplex method to get an optimal basis B.

Note: B is a basis for the equivalent LP in SEF;
i.e., B is a basis for $A' = [A \mid I]$.

To find a dual optimal solution y^*, we **normally** solve

$$(A'_B)^T y^* = c'_B$$

where c' is the objective vector for the LP in SEF;
i.e., $(c')^T = [c^T \mid 0]$.

Alternatively: We have established before that

$$c'_j - (A'_j)^T y^* = \bar{c}_j \ \ (j = 1, 2, \ldots, n+m)$$

For $j = n + i$ $(i = 1, 2, \ldots, m)$ (i.e., slack variables),

$c'_j = 0$ and A'_j is the i-th column of I.

$\Longrightarrow \quad -y^*_{n+i} = \bar{c}_{n+i}$ (or equivalently, $y^*_{n+i} = -\bar{c}_{n+i}$)

$\Longrightarrow \quad y^*_{n+i} =$ coefficient of x_{n+i} in the z-row of final optimal tableau

Example (Not in notes)

Recall the example LP.

$$\begin{array}{llrcrcrcrcrcr} \text{max.} & z = & 5x_1 & + & 3x_2 & & & & & & & & \\ \text{s.t.} & & 2x_1 & + & 3x_2 & + & x_3 & & & & & = & 15 \\ & & 2x_1 & + & x_2 & & & + & x_4 & & & = & 9 \\ & & x_1 & - & x_2 & & & & & + & x_5 & = & 3 \\ & & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq & 0 \end{array}$$

Applying the simplex method results in the final optimal tableau

$$\begin{array}{lllrrrrrrcr} z & & & + & \frac{1}{4}x_3 & + & \frac{9}{4}x_4 & & & = & 24 \\ & x_1 & & - & \frac{1}{4}x_3 & + & \frac{3}{4}x_4 & & & = & 3 \\ & & x_2 & + & \frac{1}{2}x_3 & - & \frac{1}{2}x_4 & & & = & 3 \\ & & & & \frac{3}{4}x_3 & - & \frac{5}{4}x_4 & + & x_5 & = & 3 \end{array}$$

corresponding to the optimal basis $B = \{1, 2, 5\}$.

Notice that there is an identity matrix in the original LP problem, namely $[A_3 \mid A_4 \mid A_5] = I$.

To get dual optimal solution from final optimal tableau, we take the coefficients of x_3, x_4 and x_5 in the z-row.

So a dual optimal solution is $y = [1/4,\ 9/4,\ 0]^T$.

Economic Interpretation of Duality, Revisited (§6.8)

In the duality chapter, we saw that the set of shadow prices forms a dual optimal solution.

We now show that under nondegeneracy, the "converse" is true: the dual optimal solution given by the simplex method is a set of shadow prices.

Recall the definitions:

(Defn) **Degenerate basic solution**

A basic solution x^* with $< m$ nonzero entries.

(Defn) **Shadow prices**

Fair prices on resources such that it is NOT advantageous to buy or sell small amount of the resources.

Consider the LP

$$(P) \qquad \begin{array}{rrcl} \max & c^T x & & \\ \text{s.t.} & Ax & = & b \\ & x & \geq & 0 \end{array}$$

Suppose we change one of the r.h.s. value b_p to $b_p + \varepsilon$, where ε is a small value, and get

$$(P') \qquad \begin{array}{rrcl} \max & c^T x & & \\ \text{s.t.} & Ax & = & b' \\ & x & \geq & 0 \end{array}$$

Theorem 6.1 (Pg 83). Let B be an optimal basis for (P), let x^* be the optimal basic solution it determines, and let $y^* = (A_B^T)^{-1} c_B$ be the dual optimal solution determined from B.
x^* is nondegenerate and $|\varepsilon|$ is sufficiently small
$\implies$ the optimal value of (P') is $c^T x^* + \varepsilon y_p^*$.

Moreover, B is also an optimal basis for (P').

Proof:
Let $\hat{x}$ be the basic solution of (P') determined by B;
i.e., $A\hat{x} = b'$ and $\hat{x}_j = 0$ for $j \in N$.

So $A_B \hat{x}_B = b' = b + \varepsilon e_p$, where e_p is the p-th unit vector.

$$\hat{x}_B \;=\; A_B^{-1}(b + \varepsilon e_p) \;=\; A_B^{-1} b + \varepsilon A_B^{-1} e_p \;=\; x_B^* + \varepsilon A_B^{-1} e_p$$

x^* nondegenerate $\implies$ $x_i^* > 0$ for $i \in B$.

So if $|\varepsilon|$ is small enough, $x_B^* + \varepsilon A_B^{-1} e_p > 0$, and $\hat{x}$ is feasible.

$\implies$ the tableau (T') for (P') corresponding to B is feasible.

Since the z-row in (T') is the same as the z-row in the tableau for (P) corresponding to B, (T') is optimal.
So B is an optimal basis for (P').

The optimal value of (P') is

$$\begin{aligned} c^T \hat{x} &= c_B^T \hat{x}_B = c_B^T (x_B^* + \varepsilon A_B^{-1} e_p) = c_B^T x_B^* + \varepsilon c_B^T A_B^{-1} e_p \\ &= c^T x^* + \varepsilon (y^*)^T e_p = c^T x^* + \varepsilon y_p^* \end{aligned}$$

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x^* is nondegenerate and $|\varepsilon|$ is sufficiently small
$\Longrightarrow$ the optimal value of (P') is $c^T x^* + \varepsilon y_p^*$.

Moreover, B is also an optimal basis for (P').

A few remarks: (Not in notes)

- The theorem implies

  the shadow price for resource p is y_p^*

  If resource p is worth more than y_p^*, then we should sell it since we only lose $-\varepsilon y_p^*$ for every ε units we sell.

  If resource p cost less than y_p^*, then we should buy it since we make εy_p^* extra for every ε units we buy.

- The LP may have more than one optimal basis B that determines nondegenerate solution.

  The optimal value of (P') should only depend on ε and p.

  Conclusion: the dual optimal solution $y^* = (A_B)^{-1}c_B$ does NOT depend on B!.

Duality and degeneracy (§6.9)

Theorem 6.2 (Pg 83). If (P) has an optimal basic solution x^* that is nondegenerate, then the dual problem (D) has a unique optimal solution.

Proof: We use C.S. Theorem:
"If x^* and y^* feasible for (P) and (D) respectively, then x^* and y^* optimal $\iff$ x^* and y^* satisfy C.S. condition."

Let y^* be ANY optimal solution of (D).

x^*, y^* optimal $\implies$ C.S. conditions:

$$x_j^* = 0 \quad \text{or} \quad A_j^T y^* = c_j$$

x^* nondegenerate $\implies$ $x_j^* > 0$ for $j \in B$.

So $\quad A_j^T y^* = c_j$ for $j \in B$;

i.e., $\quad A_B^T y^* = c_B$

Conclusion: Every dual optimal solution solves $A_B^T y = c_B$.

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A few remarks: (Not in notes)

- The theorem requires that x^* is **nondegenerate**.
- A_B is nonsingular **alone is not enough** to conclude that there is a unique dual solution.