CO350 Linear Programming
Chapter 6: The Simplex Method

6th June 2005

Recap

Last week, we learned the definition of a tableau and associated terminology.

Example:

$$
\begin{array}{llllllllllll}
\text{max.} & 5x_1 & + & 3x_2 & & & & & & & & \\
\text{s.t.} & 2x_1 & + & 3x_2 & + & x_3 & & & & & = & 15 \\
 & 2x_1 & + & x_2 & & & + & x_4 & & & = & 9 \\
 & x_1 & - & x_2 & & & & & + & x_5 & = & 3 \\
 & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq & 0
\end{array}
$$

Final tableau (see slides on course page):

$$
\begin{array}{llllllllll}
z & & & + & \frac{1}{4}x_3 & + & \frac{9}{4}x_4 & & & = & 24 \\
 & & & & \frac{3}{4}x_3 & - & \frac{5}{4}x_4 & + & x_5 & = & 3 \\
 & & x_2 & + & \frac{1}{2}x_3 & - & \frac{1}{2}x_4 & & & = & 3 \\
 & x_1 & & - & \frac{1}{4}x_3 & + & \frac{3}{4}x_4 & & & = & 3
\end{array}
$$

Proof of optimality

$$\begin{array}{lllllllllll} z & & & + & \frac{1}{4}x_3 & + & \frac{9}{4}x_4 & & & = & 24 \\ & & & & \frac{3}{4}x_3 & - & \frac{5}{4}x_4 & + & x_5 & = & 3 \\ & & x_2 & + & \frac{1}{2}x_3 & - & \frac{1}{2}x_4 & & & = & 3 \\ & x_1 & & - & \frac{1}{4}x_3 & + & \frac{3}{4}x_4 & & & = & 3 \end{array}$$

We can actually prove **algebraically** that the current bfs is optimal.

From the z-row: $z + \frac{1}{4}x_3 + \frac{9}{4}x_4 = 24$,

we get $z = 24 - \frac{1}{4}x_3 - \frac{9}{4}x_4$

Since $x_4, x_5 \geq 0$ for any feasible solution, we have

$$z = 24 - \frac{1}{4}x_3 - \frac{9}{4}x_4 \leq 24$$

for any feasible solution.

The solution $x^* = [3, 3, 0, 0, 3]^T$ has objective value 24, so it must an optimal solution.

We can also prove optimality using the C.S. Theorem. (More on this later.)

Working out the Details (§6.3)

Each step of the simplex method involves

1. Picking a nonbasic variable x_k to increase.
2. Determining how much to increase x_k.
3. Forming the next tableau.

1. **Choosing Entering Variable (Pg 75)**

   By picking a nonbasic x_k with $\bar{c}_k > 0$ and increase its value, we can hope to increase the objective value z.

   So we pick such nonbasic variable to join the next basis. We say that x_k enters the basis.

2. **Choosing Leaving Variable (Pg 75)**

   When we increase x_k from 0 to t while holding the other nonbasics at 0, the basic variables changes as follows

$$x_i(t) = \bar{b}_i - \bar{a}_{ik} t, \ \ (i \in B)$$

   If $\bar{a}_{ik} \leq 0$, then $\quad x_i(t) = \bar{b}_i - \bar{a}_{ik} t \geq \bar{b}_i \geq 0$ for all $t \geq 0$.

   If $\bar{a}_{ik} > 0$, then $\quad x_i(t) = \bar{b}_i - \bar{a}_{ik} t \geq 0 \iff t \leq \bar{b}_i / \bar{a}_{ik}$.

   To maintain feasibility, we need $t \leq \bar{b}_i / \bar{a}_{ik}$ for all $\bar{a}_{ik} > 0$. So we take $t = \min\{\bar{b}_i / \bar{a}_{ik} : \bar{a}_{ik} > 0\}$.

   Suppose r is the index for which $t = \bar{b}_r / \bar{a}_{rk}$; i.e., $x_r(t) = 0$ with the above choice of t.

   Now that the basic variable x_r takes value 0 in the new solution, it becomes nonbasic in the next basis. We say that x_r leaves the basis.

Working out the Details (cont'd)

Each step of the simplex method involves

1. Picking a nonbasic variable x_k to increase.
2. Determining how much to increase x_k.
3. Forming the next tableau.

3. **Pivoting (Pg 74)**

   The new basis will have x_k enters and x_r leaves;

   i.e., $\text{(new } B) = \text{(old } B) \cup \{k\} \setminus \{r\}$

   Tableau for old B:

$$
\begin{array}{lllcccccccl}
z & & & - & \cdots & - & \bar{c}_k x_k & - & \cdots & = & \bar{v} \\
& x_i & & + & \cdots & + & \bar{a}_{ik} x_k & + & \cdots & = & \bar{b}_i \quad (i \in B \setminus \{r\}) \\
& & x_r & + & \cdots & + & \bar{a}_{rk} x_k & + & \cdots & = & \bar{b}_r
\end{array}
$$

   To get tableau corresponding to new B, we "transfer" the column for x_r to x_k.

   This is called pivoting on (r, k).

   - $\text{(new } z\text{-row)} = \text{(old } z\text{-row)} - \dfrac{-\bar{c}_k}{\bar{a}_{rk}} \times \text{(old } x_r\text{-row)}$
   - for $i \in B \setminus \{r\}$,

     $\text{(new } x_i\text{-row)} = \text{(old } x_i\text{-row)} - \dfrac{\bar{a}_{ik}}{\bar{a}_{rk}} \times \text{(old } x_r\text{-row)}$
   - $\text{(new } x_k\text{-row)} = \dfrac{1}{\bar{a}_{rk}} \times \text{(old } x_r\text{-row)}$

Working out the Details (cont'd)

We terminate the simplex method when

4. we cannot pick an entering variable, or
5. we cannot pick a leaving variable.

4. **Recognizing Optimality (Pg 76)**

If $\bar{c}_k \leq 0$ for all nonbasic k, then we cannot pick an entering variable.

Proof of optimality of x^*

For any feasible solution $\hat{x}$,

$$\hat{x}_k \geq 0 \text{ and } \bar{c}_k \text{ for all } k \in N \implies \sum_{j \in N} \bar{c}_j \hat{x}_j \leq 0$$

From the z-row, $\hat{x}$ has objective value

$$z = \bar{v} + \sum_{j \in N} \bar{c}_j \hat{x}_j \leq \bar{v}$$

So the optimal value of the LP is at most $\bar{v}$.

Since x^* has value $\bar{v}$, it must be optimal. ■

In this case, the basis B the determines x^* is called an optimal basis,

and the corresponding tableau is an optimal tableau.

Working out the Details (cont'd)

We terminate the simplex method when

4. we cannot pick an entering variable, or
5. we cannot pick a leaving variable.

5. **Recognizing Unboundedness (Pg 75)**

After choosing entering variable x_k, it may happen that

$$\bar{a}_{ik} \leq 0 \text{ for all } i \in B$$

In this case, we have no upper bound on t.
(Recall upper bounds are $t \leq \bar{b}_i/\bar{a}_{ik}$ for all $\bar{a}_{ik} > 0$)
I.e., we can increase t and hence the objective value $z(t)$ without bound — the LP is unbounded.

<u>Proof of unboundedness</u> (Not in notes)

Let

$$\begin{aligned} x_i(t) &= \bar{b}_i - \bar{a}_{ik}t && \text{for } i \in B, \\ x_j(t) &= 0 && \text{for } j \in N \setminus \{k\}, \\ x_k(t) &= t. \end{aligned}$$

For $i \in B$, $\bar{a}_{ik} \leq 0 \implies x_i(t) \geq \bar{b}_i \geq 0$ for all $t \geq 0$.
For $j \in N$, we clearly have $x_j(t) \geq 0$.

Since $x(t)$ satisfies all x_i-rows in the current tableau, it must also satisfy $Ax = b$.
Thus it is feasible for $t \geq 0$.

From the z-row, $x(t)$ has objective value

$$z(t) = \bar{v} + \bar{c}_k t \to \infty \text{ as } t \to \infty.$$

So the LP is unbounded. ∎

The Simplex Method (Pg 77)

Assumptions

Given A is m-by-n with rank m, and LP problem

$$\begin{array}{lrcl} \text{maximize} & z & = & c^T x \\ \text{subject to} & Ax & = & b \\ & x & \geq & 0 \end{array}$$

Also given a feasible basis B.

Algorithm

1. Begin with feasible tableau corresponding to B.

2. If $\bar{c}_j \leq 0$ for all $j \in N$, stop. B is an optimal basis.

3. Pick $k \in N$ with $\bar{c}_k > 0$.

4. If $\bar{a}_{ik} \leq 0$ for all $i \in B$, stop. The LP is unbounded.

5. Pick $r \in B$ with $\bar{a}_{rk} > 0$ and

$$\frac{\bar{b}_r}{\bar{a}_{rk}} = \min \left\{ \frac{\bar{b}_i}{\bar{a}_{ik}} : \bar{a}_{ik} > 0 \right\}$$

6. Pivot on (r, k). Replace B by $B \cup \{k\} \setminus \{r\}$.
   Go to Step 2.

Example (Not in notes)

Solve the LP using the simplex method.

$$\begin{array}{llcllcl} \text{maximize} & -x_1 & + & 3x_2 & & \\ \text{subject to} & -x_1 & + & x_2 & \leq & 1 \\ & -x_1 & + & 2x_2 & \leq & 3 \\ & x_1 & , & x_2 & \geq & 0 \end{array}$$

Add the slack variables x_3 and x_4 to get

$$\begin{array}{llcllcllcl} \text{maximize} & -x_1 & + & 3x_2 & & & & & & \\ \text{subject to} & -x_1 & + & x_2 & + & x_3 & & & = & 1 \\ & -x_1 & + & 2x_2 & & & + & x_4 & = & 3 \\ & x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0 \end{array}$$

Clearly $B = \{3, 4\}$ is a feasible basis with corresponding tableau

$$\begin{array}{ccrcrcrcrcl} z & + & x_1 & - & 3x_2 & & & & & = & 0 \\ & & -x_1 & + & x_2 & + & x_3 & & & = & 1 \\ & & -x_1 & + & 2x_2 & & & + & x_4 & = & 3 \end{array}$$

$\bar{c}_2 = 3 > 0$, so x_2 enters.

$t = \min\{1/1, 3/2\} = 1$, so x_3 leaves.

Pivot on $(3, 2)$ to get tableau corresponding to $B = \{2, 4\}$.

$$\begin{array}{rrcrcrcrcl}
(z\text{-row}) + 3 \times (x_3\text{-row}): & z - & 2x_1 & & & + 3x_3 & & & = & 3 \\
(x_3\text{-row}): & & -x_1 & + & x_2 & + \;\; x_3 & & & = & 1 \\
(x_4\text{-row}) - 2 \times (x_3\text{-row}): & & x_1 & & & - \; 2x_3 & + & x_4 & = & 1
\end{array}$$

$\bar{c}_1 = 2 > 0$, so x_1 enters.
$t = \min\{-, 1/1\} = 1$, so x_4 leaves.
Pivot on $(4, 1)$ to get tableau corresponding to $B = \{1, 2\}$.

$$\begin{array}{rrcrcrcrcl}
(z\text{-row}) + 2 \times (x_4\text{-row}): & z & & & - & x_3 & + & 2x_4 & = & 5 \\
(x_2\text{-row}) + (x_4\text{-row}): & & & x_2 & - & x_3 & + & x_4 & = & 2 \\
(x_4\text{-row}): & & x_1 & & - & 2x_3 & + & x_4 & = & 1
\end{array}$$

$\bar{c}_3 = 1 > 0$, so x_3 enters.
$a_{i3} \leq 0$ for all $i \in B = \{1, 2\}$, so the LP is unbounded.

<u>Proof of unboundedness</u>

Let $x_1(t) = 1 + 2t$, $x_2(t) = 2 + t$, $x_3(t) = t$, $x_4(t) = 0$.
Clearly $x(t) \geq 0$ for all $t \geq 0$.
Also $-x_1(t) + x_2(t) + x_3(t) = -(1 + 2t) + (2 + t) + t = 1$,
and $-x_1(t) + 2x_2(t) + x_4(t) = -(1 + 2t) + 2(2 + t) = 3$.
Thus $x(t)$ is feasible for all $t \geq 0$.
$x(t)$ has objective value

$$-x_1(t) + 3x_2(t) = -(1 + 2t) + 3(2 + t) = 5 + t \to \infty$$

as $t \to \infty$. So the LP is unbounded. ■