# CO350 Linear Programming
# Chapter 6: The Simplex Method

3rd June 2005

# Recap

Suppose $A$ is an $m$-by-$n$ matrix with rank $m$.

$$(P) \qquad \begin{array}{lrcl} \text{max.} & c^T x & & \\ \text{s.t.} & Ax & = & b \\ & x & \geq & 0 \end{array}$$

On Wednesday, we learned

- if $(P)$ has an optimal solution, then it has an optimal solution that is basic.
- the motivation for simplex method: we search for an optimal solution one bfs at a time.
- how to obtain the basic solution determined by a given basis, while preserving the data of the LP.

# Bases and Tableaux

Suppose $A$ is $m$-by-$n$ with rank $m$.

$$(P) \qquad \begin{array}{lrcl} \text{max.} & (z \; =) & c^T x & \\ \text{s.t.} & & Ax & = & b \\ & & x & \geq & 0 \end{array}$$

(Defn) **Tableau**

A tableau for $(P)$ corresponding to a basis $B$ is the system of equations of the form

$z$-row $\longrightarrow$ $$z - \sum_{j \in N} \bar{c}_j x_j = \bar{v}$$

$x_i$-row $\longrightarrow$ $$x_i + \sum_{j \in N} \bar{a}_{ij} x_j = \bar{b}_i \qquad (i \in B)$$

derived by applying elementary row operations to

$$\begin{array}{rcl} z - c^T x & = & 0 \\ Ax & = & b \end{array}$$

Example $(B = \{1, 2, 3\})$

$$\begin{array}{llllllll} z & & & + \; 8/3\, x_4 & - \; 1/3\, x_5 & = & 23 & (z\text{-row}) \\ & & x_3 & - \; 5/3\, x_4 & + \; 4/3\, x_5 & = & 4 & (x_3\text{-row}) \\ x_1 & & & + \; 1/3\, x_4 & + \; 1/3\, x_5 & = & 4 & (x_1\text{-row}) \\ & x_2 & & + \; 1/3\, x_4 & - \; 2/3\, x_5 & = & 1 & (x_2\text{-row}) \end{array}$$

$\bar{c}_4 = -8/3$ $\qquad$ $\bar{c}_5 = 1/3$ $\qquad$ $\bar{v} = 23$

$\bar{a}_{34} = -5/3$ $\qquad$ $\bar{a}_{25} = -2/3$ $\qquad$ $\bar{b}_2 = 1$

## A tableau has a wealth of information (Pg 71)

Tableau corresponding to $B = \{1, 2, 3\}$:

$$\begin{array}{lllllllllll}
z & & & & + & 8/3\ x_4 & - & 1/3\ x_5 & = & 23 \\
 & & & x_3 & - & 5/3\ x_4 & + & 4/3\ x_5 & = & 4 \\
 & x_1 & & & + & 1/3\ x_4 & + & 1/3\ x_5 & = & 4 \\
 & & x_2 & & + & 1/3\ x_4 & - & 2/3\ x_5 & = & 1
\end{array}$$

1. The basic solution determined by $B = \{1, 2, 3\}$ is
$$x^* = [4, 1, 4, 0, 0]^T$$
   In general, the basic solution determined by $B$ is
$$x_i^* = \bar{b}_i \text{ for } i \in B$$
$$x_j^* = 0 \text{ for } j \in N$$
   The solution is a bfs if and only if $\bar{b}_i \geq 0$ for all $i \in B$.

   In this case, we say that the tableau is feasible, and $B$ is a feasible basis.

   E.g., $B = \{1, 2, 3\}$ is feasible but $B = \{2, 3, 5\}$ is not feasible.

2. The objective value of the basic solution determined by $B = \{1, 2, 3\}$ is $z = \bar{v} = 23$

   In general, the objective value is given by $\bar{v}$.

## A tableau has a wealth of information (cont'd)

Tableau corresponding to $B = \{1, 2, 3\}$:

$$\begin{array}{rrrrrrrrrr} z & & & & + & 8/3\ x_4 & - & 1/3\ x_5 & = & 23 \\ & & & x_3 & - & 5/3\ x_4 & + & 4/3\ x_5 & = & 4 \\ & x_1 & & & + & 1/3\ x_4 & + & 1/3\ x_5 & = & 4 \\ & & x_2 & & + & 1/3\ x_4 & - & 2/3\ x_5 & = & 1 \end{array}$$

3. When we increase $x_5$ from $0$ while keeping the other nonbasic variable $x_4$ at $0$, we see that the term $\boxed{-1/3\ x_5}$ in the $z$-row decreases.

   So $z$ have to increase to counter this.
   Note: $\bar{c}_5 = 1/3\ > 0$.

   When we increase $x_4$ from $0$ while keeping the other nonbasic variable $x_5$ at $0$, we see that the term $\boxed{8/3\ x_4}$ in the $z$-row increase.

   So $z$ have to decreases to counter this.
   Note: $\bar{c}_4 = -8/3\ < 0$.

   $\bar{c}_k > 0 \implies$ increasing $x_k$ increases $z$.

   $\bar{c}_k < 0 \implies$ increasing $x_k$ decreases $z$.

   $\bar{c}_k = 0 \implies$ increasing $x_k$ does not change $z$.

   $\bar{c}_k$ is called the reduced cost of the nonbasic $x_k$.

# The Idea of the Simplex Method

Illustration with an example

(This illustrates the idea of the method, and not how the simplex method should be applied.)

Recall the example LP.

$$
\begin{array}{lrcrcrcrcrcrcl}
\text{max.} & z = & 5x_1 & + & 3x_2 & & & & & & & & \\
\text{s.t.} & & 2x_1 & + & 3x_2 & + & x_3 & & & & & = & 15 \\
 & & 2x_1 & + & x_2 & & & + & x_4 & & & = & 9 \\
 & & x_1 & - & x_2 & & & & & + & x_5 & = & 3 \\
 & & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq & 0
\end{array}
$$

The tableau corresponding to the basis $B = \{3, 4, 5\}$ is

$$
\begin{array}{rcrcrcrcrcrcl}
z & - & 5x_1 & - & 3x_2 & & & & & & & = & 0 \\
 & & 2x_1 & + & 3x_2 & + & x_3 & & & & & = & 15 \\
 & & 2x_1 & + & x_2 & & & + & x_4 & & & = & 9 \\
 & & x_1 & - & x_2 & & & & & + & x_5 & = & 3
\end{array}
$$

and it determines the bfs $x^* = [0, 0, 15, 9, 3]^T$.

We saw that for a nonbasic variable $x_k$ that has reduced cost $\bar{c}_k > 0$, increasing $x_k$ from $0$ while keeping the other nonbasic variables (in this case there is only one other, but in general there could be many more) at $0$ will increase the value of $z$ (i.e., the objective value).

In the tableau, the nonbasic variable $x_1$ has reduced cost $\bar{c}_1 = 5 > 0$ (we can use $x_2$ instead since $\bar{c}_2 = 3 > 0$).

Increasing $x_1$ to $t \geq 0$ while keeping $x_2$ at $0$ gives

$$\begin{aligned} z(t) &= 0 + 5t \\ x_1(t) &= t \\ x_2(t) &= 0 \\ x_3(t) &= 15 - 2t \\ x_4(t) &= 9 - 2t \\ x_5(t) &= 3 - t \end{aligned}$$

From the expression for $z(t)$ we know that the larger $t$ is, the better our solution is. BUT we need to make sure our solution is feasible. Maintaining feasibility means

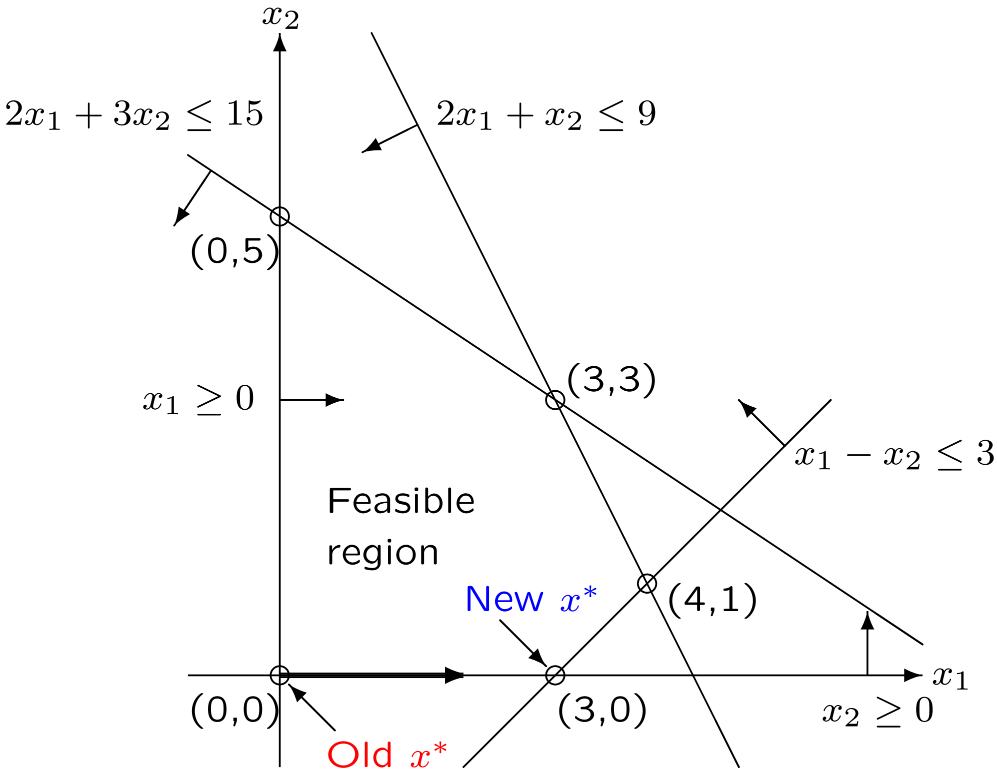
$$\left.\begin{aligned} 15 - 2t &\geq 0 \quad \text{which implies } t \leq 7.5 \\ 9 - 2t &\geq 0 \quad \text{which implies } t \leq 4.5 \\ 3 - t &\geq 0 \quad \text{which implies } t \leq 3 \end{aligned}\right\} \implies t \leq 3.$$

So we use $t = 3$, which gives the new feasible solution $x^* = [3, 0, 9, 3, 0]^T$ with objective value $15$.

Note that this is again a bfs! This is NOT a coincidence.

This bfs is determined by the basis $B = \{1, 3, 4\}$.

Here is a diagram of the problem in SIF (treating $x_3$, $x_4$, $x_5$ as slacks).




The solution $x(t) = [x_1(t), x_2(t), \ldots, x_5(t)]^T$ is given by

$$\left.\begin{array}{rcl} x_1(t) &=& t \\ x_2(t) &=& 0 \\ x_3(t) &=& 15 - 2t \\ x_4(t) &=& 9 - 2t \\ x_5(t) &=& 3 - t \end{array}\right\} \Longrightarrow x(t) = \begin{bmatrix} 0 \\ 0 \\ 15 \\ 9 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -2 \\ -2 \\ -1 \end{bmatrix}$$

i.e., in the space of all 5 variables, we move from $[0, 0, 15, 9, 3]^T$ in the direction of $[1, 0, -2, -2, -1]^T$.

In the space of $[x_1, x_2]^T$, we move from $[0, 0]^T$ in the direction of $[1, 0]^T$.

We shall get the tableau corresponding to $B = \{1,3,4\}$ from

$$\begin{array}{rcrcrcrcrcrcr} z & - & 5x_1 & - & 3x_2 & & & & & & & = & 0 \\ & & 2x_1 & + & 3x_2 & + & x_3 & & & & & = & 15 \\ & & 2x_1 & + & x_2 & & & + & x_4 & & & = & 9 \\ & & x_1 & - & x_2 & & & & & + & x_5 & = & 3 \end{array}$$

Note that the columns for $x_3$ and $x_4$ (previous basic variables that remain basic) already form part of the identity matrix (so does the column for $z$, but it always does).

So we only need to make the column for $x_1$ (a new member of the basic variables) the remaining column of the identity matrix.

In a way, we are "transferring" the column for $x_5$ to $x_1$.

Good news: This can be done with at most four elementary row operations – at most one on each row.

$$\begin{array}{rl} (z\text{-row}) + 5 \times (x_5\text{-row}): & z \qquad - 8x_2 \qquad\qquad\qquad + 5x_5 = 15 \\ (x_3\text{-row}) - 2 \times (x_5\text{-row}): & \qquad\quad 5x_2 + x_3 \qquad\quad - 2x_5 = 9 \\ (x_4\text{-row}) - 2 \times (x_5\text{-row}): & \qquad\quad 3x_2 \qquad + x_4 - 2x_5 = 3 \\ (x_5\text{-row}): & \quad x_1 - x_2 \qquad\qquad\quad + x_5 = 3 \end{array}$$

By changing only one member of $B$, we need at most $m+1$ row operations to get the new tableau.

In the new tableau, the nonbasic variable $x_2$ has reduced cost $\bar{c}_2 = 8 > 0$ (we cannot use $x_5$ since $\bar{c}_3 = -5 \not> 0$).

Increasing $x_2$ to $t \geq 0$ while keeping $x_5$ at $0$ gives

$$\begin{aligned} z(t) &= 15 + 8t \\ x_1(t) &= 3 + t \\ x_2(t) &= t \\ x_3(t) &= 9 - 5t \\ x_4(t) &= 3 - 3t \\ x_5(t) &= 0 \end{aligned}$$

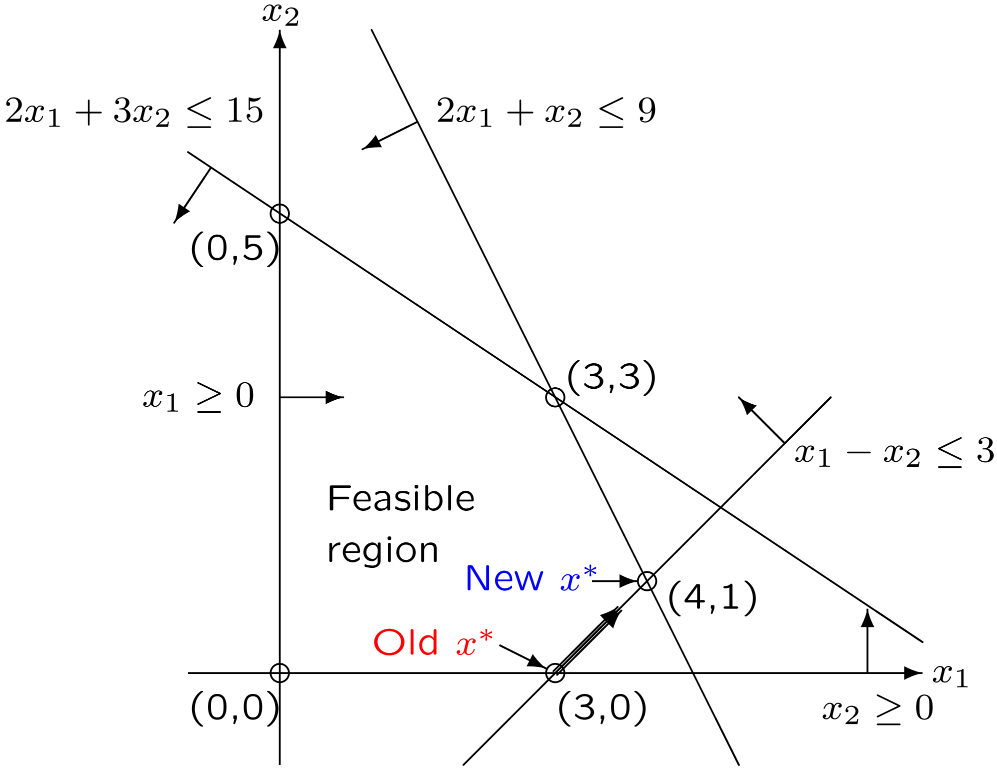
Once again the larger $t$ is, the better our solution is. To maintain feasibility, we need

$$\left.\begin{array}{rcll} 3 + t & \geq & 0 & \text{which implies no upper bound on } t \\ 9 - 5t & \geq & 0 & \text{which implies } t \leq 1.8 \\ 3 - 3t & \geq & 0 & \text{which implies } t \leq 1 \end{array}\right\} \Longrightarrow t \leq 1.$$

So we use $t = 1$, which gives the new feasible solution $x^* = [4, 1, 4, 0, 0]^T$ with objective value $23$.

Once again, this is a bfs, and it is determined by the basis $B = \{1, 2, 3\}$.

The following diagram shows this step.




The solution $x(t) = [x_1(t), x_2(t), \ldots, x_5(t)]^T$ is given by

$$\left.\begin{array}{rcl} x_1(t) & = & 3+t \\ x_2(t) & = & t \\ x_3(t) & = & 9-5t \\ x_4(t) & = & 3-3t \\ x_5(t) & = & 0 \end{array}\right\} \implies x(t) = \begin{bmatrix} 3 \\ 0 \\ 9 \\ 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ -5 \\ -3 \\ 0 \end{bmatrix}$$

i.e., in the space of all $5$ variables, we move from $[3,0,9,3,0]^T$ in the direction of $[1,1,-5,-3,0]^T$.

In the space of $[x_1, x_2]^T$, we move from $[3,0]^T$ in the direction of $[1,1]^T$.

To get the tableau corresponding to $B = \{1, 2, 3\}$ from

$$\begin{array}{rcrcrcrcrcrcr} z & & & - & 8x_2 & & & & & + & 5x_5 & = & 15 \\ & & & & 5x_2 & + & x_3 & & & - & 2x_5 & = & 9 \\ & & & & 3x_2 & & & + & x_4 & - & 2x_5 & = & 3 \\ & & x_1 & - & x_2 & & & & & + & x_5 & = & 3 \end{array}$$

we "transfer" the column for $x_4$ to $x_2$.

Once again we do this with at most four elementary row operations.

$$\begin{array}{rl} (z\text{-row}) + \frac{8}{3} \times (x_4\text{-row}): & z \quad + \frac{8}{3}x_4 - \frac{1}{3}x_5 = 23 \\ (x_3\text{-row}) - \frac{5}{3} \times (x_4\text{-row}): & x_3 - \frac{5}{3}x_4 + \frac{4}{3}x_5 = 4 \\ \frac{1}{3} \times (x_4\text{-row}): & x_2 \quad + \frac{1}{3}x_4 - \frac{2}{3}x_5 = 1 \\ (x_1\text{-row}) + \frac{1}{3} \times (x_4\text{-row}): & x_1 \quad + \frac{1}{3}x_4 + \frac{1}{3}x_5 = 4 \end{array}$$

We can still increase the objective value since $\bar{c}_5 = \frac{1}{3} > 0$.

Increasing $x_5$ to $t \geq 0$ while keeping $x_4$ at 0 gives

$$\begin{array}{rcl} z(t) & = & 23 + \frac{1}{3}t \\ x_1(t) & = & 4 - \frac{1}{3}t \\ x_2(t) & = & 1 + \frac{2}{3}t \\ x_3(t) & = & 4 - \frac{4}{3}t \\ x_4(t) & = & 0 \\ x_5(t) & = & t \end{array}$$

To maintain feasibility, we need

$$\left.\begin{array}{rcll} 4-\frac{1}{3}t & \geq & 0 & \text{which implies } t \leq 12 \\ 1+\frac{2}{3}t & \geq & 0 & \text{which implies no upper} \\ & & & \text{bound on } t \\ 4-\frac{4}{3}t & \geq & 0 & \text{which implies } t \leq 3 \end{array}\right\} \implies t \leq 3.$$

So we use $t = 3$, which gives the new feasible solution $x^* = [3,3,0,0,3]^T$ with objective value $24$.

This bfs is determined by the basis $B = \{1,2,5\}$.

To get the tableau corresponding to $B = \{1,2,5\}$ from

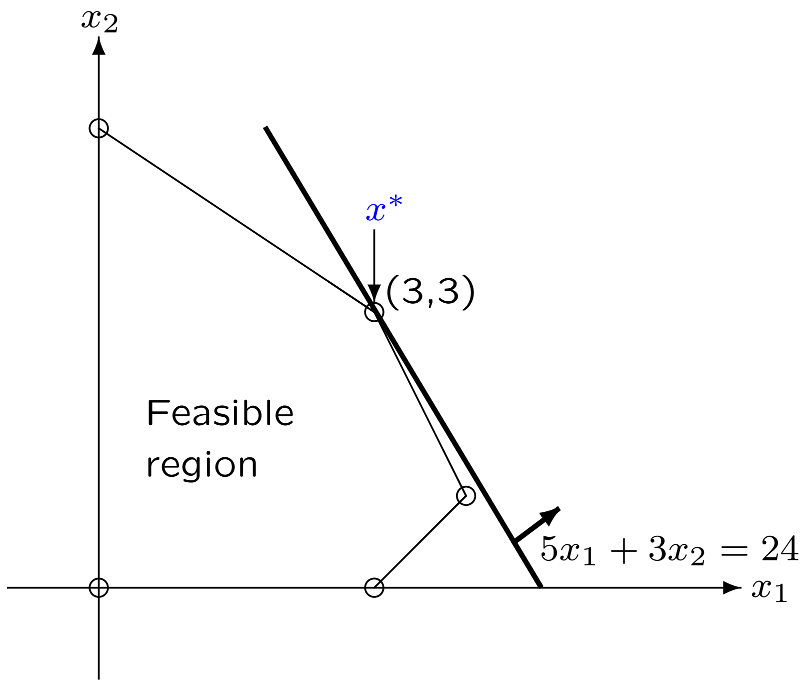
$$\begin{array}{cccccccccc} z & & & & + & \frac{8}{3}x_4 & - & \frac{1}{3}x_5 & = & 23 \\ & & & x_3 & - & \frac{5}{3}x_4 & + & \frac{4}{3}x_5 & = & 4 \\ & & x_2 & & + & \frac{1}{3}x_4 & - & \frac{2}{3}x_5 & = & 1 \\ & x_1 & & & + & \frac{1}{3}x_4 & + & \frac{1}{3}x_5 & = & 4 \end{array}$$

we "transfer" the column for $x_3$ to $x_5$, giving

$$\begin{array}{ccccccccccc} z & & & + & \frac{1}{4}x_3 & + & \frac{9}{4}x_4 & & & = & 24 \\ & & & & \frac{3}{4}x_3 & - & \frac{5}{4}x_4 & + & x_5 & = & 3 \\ & & x_2 & + & \frac{1}{2}x_3 & - & \frac{1}{2}x_4 & & & = & 3 \\ & x_1 & & - & \frac{1}{4}x_3 & + & \frac{3}{4}x_4 & & & = & 3 \end{array}$$

Now we can no longer proceed since both reduced costs $\bar{c}_3 = -1/4$ and $\bar{c}_4 = -9/4$ are not positive.

The diagram below shows that $x^*$ is optimal.




We can actually prove it algebraically.

From the $z$-row: $z + \frac{1}{4}x_3 + \frac{9}{4}x_4 = 24$,

we get
$$z = 24 - \frac{1}{4}x_3 - \frac{9}{4}x_4$$

Since $x_4, x_5 \geq 0$ for any feasible solution, we have
$$z = 24 - \frac{1}{4}x_3 - \frac{9}{4}x_4 \leq 24$$
for any feasible solution.

The solution $x^* = [3, 3, 0, 0, 3]^T$ has objective value 24, so it must an optimal solution.

We can also prove optimality using the C.S. Theorem. (More on this later.)