CO350 Linear Programming
Chapter 5: Basic Solutions

1st June 2005

Recap

On Monday, we learned

- **Theorem 5.3**
  Consider an LP in SEF with $rank(A) = \#$ rows. Then
  x^* is bfs $\iff$ x^* is extreme point of the feasible region.
- Definition of basic feasible solution for LP problems in SIF.
- **Theorem 5.4**
  Consider an LP in SIF. Then
  x^* is bfs $\iff$ x^* is extreme point of the feasible region.

Why consider basic feasible solutions?

Theorem 5.5 (Pg 65) Let A be m by n with rank m. Consider the LP in SEF

$$(P) \qquad \begin{array}{lrcl} \text{max.} & c^T x & & \\ \text{s.t.} & Ax & = & b \\ & x & \geq & 0 \end{array}$$

If (P) has an optimal solution

then (P) has an optimal solution that is basic.

Proof: (Almost the same as the proof on page 66)

> Key ingredient:
> Show that if x^* is optimal but not basic, then there is an optimal solution with more zeros entries than x^*.

Suppose that x^* is optimal but not basic.

From proof of previous theorem (also emphasized in the remarks after proof), we know that
$\exists d \in \mathbf{R}^n$, $d \neq 0$, $d_j = 0$ whenever $x^*_j = 0$, and $Ad = 0$, and
$\exists \varepsilon > 0$, both $x^1 = x^* + \varepsilon d$ and $x^2 = x^* - \varepsilon d$ are feasible.

We now show that $c^T d = 0$.

If $c^T d > 0$, then

$$c^T x^1 = c^T x^* + \varepsilon c^T d > c^T x^*$$

contradicts x^* is optimal.

Similarly, $c^T d < 0 \implies c^T x^2 > c^T x^*$ contradicts x^* optimal.

Proof: (cont'd)

Key ingredient:
Show that if x^* is optimal but not basic, then there is an optimal solution with more zeros entries than x^*.

So far we assumed x^* optimal but not basic and obtained non-zero vector $d \in \mathbf{R}^n$ such that

(1) $Ad = 0$, $c^T d = 0$, and

(2) $d_j = 0$ whenever $x_j^* = 0$.

Since d has at least one non-zero component, we may assume that $d_k < 0$ for some k. (Consider $-d$ otherwise.)

Let $x(t) = x^* + td$. Note that $Ax(t) = Ax^* + tAd = b$.

As t increases from 0, at least one component of $x(t)$ decreases towards 0. (E.g., the k-th component.)

So there is a largest value of t for which $x(t)$ stays feasible.

At that largest value of t, at least one component of $x(t)$ went from positive to 0.

At the same time, all those components that were 0 remains at 0 (because of (2)).

So $x(t)$ is a feasible solution with more zero components than x^*.

Moreover, $c^T x(t) = c^T x^* + tc^T d = c^T x^* \implies x(t)$ optimal.

Conclusion: an optimal solution x^* with most possible number of zero component must be basic. ■

Theorem 5.5 (Pg 65) Let A be m by n with rank m. Consider the LP in SEF

$$(P) \qquad \begin{array}{lrcl} \text{max.} & c^T x & & \\ \text{s.t.} & Ax & = & b \\ & x & \geq & 0 \end{array}$$

If $\quad (P)$ has an optimal solution

then $\quad (P)$ has an optimal solution that is basic.

A few remarks: (Not in notes)

- The assumption **"$\boldsymbol{A}$ has rank $\boldsymbol{m}$"** is necessary!

  Without it, we cannot even define a basis of A.

  In this case, the theorem fails because (P) may still have optimal solutions but there are no basic solutions.

- The theorem does **NOT** say that all optimal solutions are basic.

  However, there is a very special case for which all optimal solutions are basic.

  All optimal solutions are basic if and only if there is exactly one optimal solution. (Prove it!)

CO350 Linear Programming
Chapter 6: The Simplex Method

1st June 2005

Motivation

Consider LP in SEF where A is m-by-n with rank m.

$$(P) \qquad \begin{array}{lrcl} \text{max.} & c^T x & & \\ \text{s.t.} & Ax & = & b \\ & x & \geq & 0 \end{array}$$

- We know that if (P) has an optimal solution, then there is one which is basic.
- So we only need to look at basic feasible solutions.
- We start at a bfs x^*, and check if x^* is optimal. (Recall C.S. Theorem!)
- If x^* is not optimal, we move to a "neighbouring" bfs that has better objective value, and repeat.

A few details to fill in

1. How do we find a bfs x^* to start with?
2. How do we move to a "neighbouring" bfs?
3. How to we know that we will find an optimal x^* after a finite number of repetitions?

For now, we consider the second question, and leave the remaining questions for later.

Bases and Tableaux

Consider LP in SEF where A is m-by-n with rank m.

$$(P) \qquad \begin{array}{lrcl} \text{max.} & c^T x & & \\ \text{s.t.} & Ax & = & b \\ & x & \geq & 0 \end{array}$$

$\boxed{\text{In this chapter, we assume that we have a basis } B \text{ that determines a bfs } x^*.}$

Tableau – "an effective way to store ALL information about the LP problem at a basis B".

(Defn) **Basic and Nonbasic Variables**

Let N be the set $\{1, 2, \ldots, n\} \setminus B$.

x_j $(j \in B)$ are called basic variables.

x_j $(j \in N)$ are called nonbasic variables.

To find the basic solution determined by B, we need to solve

$$Ax = b$$
$$x_j = 0 \quad (j \notin B)$$

To retain information about original LP, we

1. express basic variables in terms of nonbasic variables,
2. set nonbasic variables to zeros.

Example (See pg 70 for more examples)

$$\begin{array}{llllllllllllll}
\text{max.} & z & = & 5x_1 & + & 3x_2 & & & & & & & & \\
\text{s.t.} & & & 2x_1 & + & 3x_2 & + & x_3 & & & & & = & 15 \\
 & & & 2x_1 & + & x_2 & & & + & x_4 & & & = & 9 \\
 & & & x_1 & - & x_2 & & & & & + & x_5 & = & 3 \\
 & & & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq & 0
\end{array}$$

For the basis $B = \{3, 4, 5\}$, we already have the basic variables x_3, x_4, x_5 in terms of the nonbasic variables x_1, x_2.

$$\begin{array}{lllllllllllll}
z & - & 5x_1 & - & 3x_2 & & & & & & & = & 0 \\
 & & 2x_1 & + & 3x_2 & + & x_3 & & & & & = & 15 \\
 & & 2x_1 & + & x_2 & & & + & x_4 & & & = & 9 \\
 & & x_1 & - & x_2 & & & & & + & x_5 & = & 3
\end{array}$$

For the basis $B = \{1, 2, 3\}$, we perform elementary row operations to get the basics in terms of the nonbasics.

$$\begin{array}{lllllllll}
z & & & & + & \frac{8}{3}x_4 & - & \frac{1}{3}x_5 & = & 23 \\
 & & x_2 & & + & \frac{1}{3}x_4 & - & \frac{2}{3}x_5 & = & 1 \\
 & x_1 & & & + & \frac{1}{3}x_4 & + & \frac{1}{3}x_5 & = & 4 \\
 & & & x_3 & - & \frac{5}{3}x_4 & + & \frac{4}{3}x_5 & = & 4
\end{array}$$

For the basis $B = \{2, 3, 5\}$, we perform elementary row operations to get

$$\begin{array}{rcrcrcrcrcrcr}
z & + & x_1 & & & & & + & 3x_4 & & & = & 27 \\
& & 2x_1 & + & x_2 & & & + & x_4 & & & = & 9 \\
& & -4x_1 & & & + & x_3 & - & 3x_4 & & & = & -12 \\
& & 3x_1 & & & & & + & x_4 & + & x_5 & = & 12
\end{array}$$

For the basis $B = \{1, 2, 5\}$, we perform elementary row operations to get

$$\begin{array}{rcrcrcrcrcrcr}
z & & & & & + & \frac{1}{4}x_3 & + & \frac{9}{4}x_4 & & & = & 24 \\
& & & & x_2 & + & \frac{1}{2}x_3 & - & \frac{1}{2}x_4 & & & = & 3 \\
& & x_1 & & & - & \frac{1}{4}x_3 & + & \frac{3}{4}x_4 & & & = & 3 \\
& & & & & & \frac{3}{4}x_3 & - & \frac{5}{4}x_4 & + & x_5 & = & 3
\end{array}$$

For the basis $B = \{2, 4, 5\}$, we perform elementary row operations to get

$$\begin{array}{rcrcrcrcrcrcr}
z & - & 3x_1 & & & + & x_3 & & & & & = & 15 \\
& & \frac{2}{3}x_1 & + & x_2 & + & \frac{1}{3}x_3 & & & & & = & 5 \\
& & \frac{4}{3}x_1 & & & - & \frac{1}{3}x_3 & + & x_4 & & & = & 4 \\
& & \frac{5}{3}x_1 & & & + & \frac{1}{3}x_3 & & & + & x_5 & = & 8
\end{array}$$