CO350 Linear Programming Chapter 5: Basic Solutions

27th May 2005

Notation

$$(P) \qquad \begin{array}{lrcl} \text{maximize} & c^T x & & \\ \text{subject to} & Ax & = & b \\ & x & \geq & 0 \end{array}$$

Many times, we will assume that A has rank = # rows.

This is w.l.o.g.:

Apply Gaussian Elimination to $[A|b]$ and either

- $\star$ conclude $Ax = b$ has no solution, or
- $\star$ eliminate redundant row to get $A'x = b'$ where A' has rank = # rows.

Let A_j denote column j of the matrix A

A_B denote the submatrix $[A_j : j \in B]$ of the matrix A

(Def<u>n</u>) **Basis (Pg 59)**
(Do NOT confused with basis of vector space)
A subset B of $\{1, 2, \ldots, n\}$ such that

(a) $|B| = m$ (i.e., B has m elements), and

(b) A_B is nonsingular (i.e., invertible).

Note that B is a basis of A if and only if columns of A_B forms a basis of the vector space $\mathbf{R}^m$.

Example (NOT in notes)

$$A = \begin{bmatrix} 2 & 0 & -4 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

$B = \{1, 2, 3\}$ is a basis as $A_B = \begin{bmatrix} 2 & 0 & -4 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ is nonsingular.

$B = \{1, 3, 4\}$ is a basis as $A_B = \begin{bmatrix} 2 & -4 & 2 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ is nonsingular.

$B = \{1, 2, 4\}$ is NOT a basis as $A_B = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ is singular.

Suppose B is a basis for A.

Consider the system of n equations in n unknowns.

$$Ax = b$$
$$x_j = 0 \ (j \notin B)$$

This system has a unique solution. **(Why?)**

(Defn) **Basic solution determined by a basis B**
The solution to the above system of equations.

(Defn) **Basic solution of $Ax = b$**
The basic solution determined by some basis B.

Note: A basic solution always have at least $n - m$ zeros since
$$x_j = 0 \ (j \notin B)$$
I.e., a basic solution always have at most m non-zeros.

Example (NOT in notes)

Consider $Ax = b$, where

$$A = \begin{bmatrix} 2 & 0 & -4 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$$

The basic solution determined by $B = \{1, 2, 3\}$ is $[3, 1, 1, 0]^T$.

The basic solution determined by $B = \{1, 3, 4\}$ is $[2, 0, 1, 1]^T$.

Question (Similar to question on Pg 61)

$$A = \begin{bmatrix} 2 & 4 & -4 & 2 & 4 \\ 0 & 2 & 0 & 1 & -1 \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Which of the following are basic solutions of $Ax = b$?

1. $[5, 0, 1, 0, -1]^T$
2. $[3, 0, 0, 0, -1]^T$
3. $[0, 1, 0, -1, 0]^T$
4. $[0, 0, 0, 1, 0]^T$

1. First check that $[5, 0, 1, 0, -1]^T$ is a solution.
   A basic solution should have at least
   $$n - m = 5 - 2 = 3 \text{ zeros.}$$
   This solution only have 2 zeros
   $\implies$ $[5, 0, 1, 0, -1]^T$ is NOT a basic solution.

2. Check: $[3, 0, 0, 0, -1]^T$ is a solution.
   It has exactly 3 zeros. B must be $\{1, 5\}$.
   $A_B = \begin{bmatrix} 2 & 4 \\ 0 & -1 \end{bmatrix}$ is nonsingular
   $\implies$ $B = \{1, 5\}$ is a basis.
   So $[3, 0, 0, 0, -1]^T$ is the basic solution determined by $B = \{1, 5\}$.

Question (cont'd)

$$A = \begin{bmatrix} 2 & 4 & -4 & 2 & 4 \\ 0 & 2 & 0 & 1 & -1 \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Which of the following are basic solutions of $Ax = b$?

1. $[5, 0, 1, 0, -1]^T$
2. $[3, 0, 0, 0, -1]^T$
3. $[0, 1, 0, -1, 0]^T$
4. $[0, 0, 0, 1, 0]^T$

3. Check: $[0, 1, 0, -1, 0]^T$ is a solution.
   It has exactly 3 zeros. B must be $\{2, 4\}$.
   $A_B = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$ is singular $\Longrightarrow$ $B = \{2, 4\}$ is not a basis.
   So $[0, 1, 0, -1, 0]^T$ is NOT a basic solution.

4. Check: $[0, 0, 0, 1, 0]^T$ is a solution.
   It has 4 zeros. B can be $\{1, 4\}$, $\{2, 4\}$, $\{3, 4\}$ or $\{4, 5\}$.
   For $B = \{1, 4\}$, $\{3, 4\}$ or $\{4, 5\}$, A_B is nonsingular
   $\Longrightarrow$ $\{1, 4\}$, $\{3, 4\}$ and $\{4, 5\}$ are bases.
   For $B = \{2, 4\}$, A_B is singular $\Longrightarrow$ $\{2, 4\}$ is not a basis.
   So $[0, 0, 0, 1, 0]^T$ is the basic solution determined by $\{1, 4\}$, $\{3, 4\}$ and $\{4, 5\}$.

NOTE: The number of non-zeros may be less than m and a basic solution may be determined by more than one basis.

Theorem 5.1 (Pg 61) Suppose A is m-by-n with rank m, and x^* is a solution of $Ax = b$.

$$x^* \text{ is a basic solution of } Ax = b$$
$$\iff \quad \{A_j : x_j^* \neq 0\} \text{ is a linearly independent set.}$$

Proof:

"$\Longrightarrow$": Suppose that x^* is a basic solution of $Ax = b$.

x^* basic solution $\implies$ x^* is determined by some basis B.

Since $j \notin B \implies x_j^* = 0$, we have $\{j : x_j^* \neq 0\} \subseteq B$.

This means $\{A_j : x_j^* \neq 0\} \subseteq \{A_j : j \in B\}$.

Since the submatrix $A_B = [A_j : j \in B]$ is nonsingular, the set $\{A_j : j \in B\}$ must be linearly independent.

Hence $\{A_j : x_j^* \neq 0\}$ must also be linearly independent.

"$\Longleftarrow$": Suppose that $\{A_j : x_j^* \neq 0\}$ is linearly independent.

We can always extend $\{A_j : x_j^* \neq 0\}$ to form a basis of $\mathbf{R}^m$.

Let $\{A_j : j \in B\}$ be the said extension.
So $\{j : x_j^* \neq 0\} \subseteq B$ and B is a basis of A.

Since $\{j : x_j^* \neq 0\} \subseteq B$, we have $j \notin B \implies x_j^* = 0$.

Thus x^* satisfies $Ax = b$ and $x_j = 0 (j \notin B)$;
i.e., x^* is the basic solution determined by B. ■

Degeneracy

In the above example, having $< m$ non-zeros results in more than one basis determining the same basic solution.

In the proof of Theorem 5.1, having $< m$ non-zeros results in the need to extend $\{A_j : x_j^* \neq 0\}$ to form a basis.

In general, having $< m$ non-zeros in a basic solution makes things a bit more complicated.

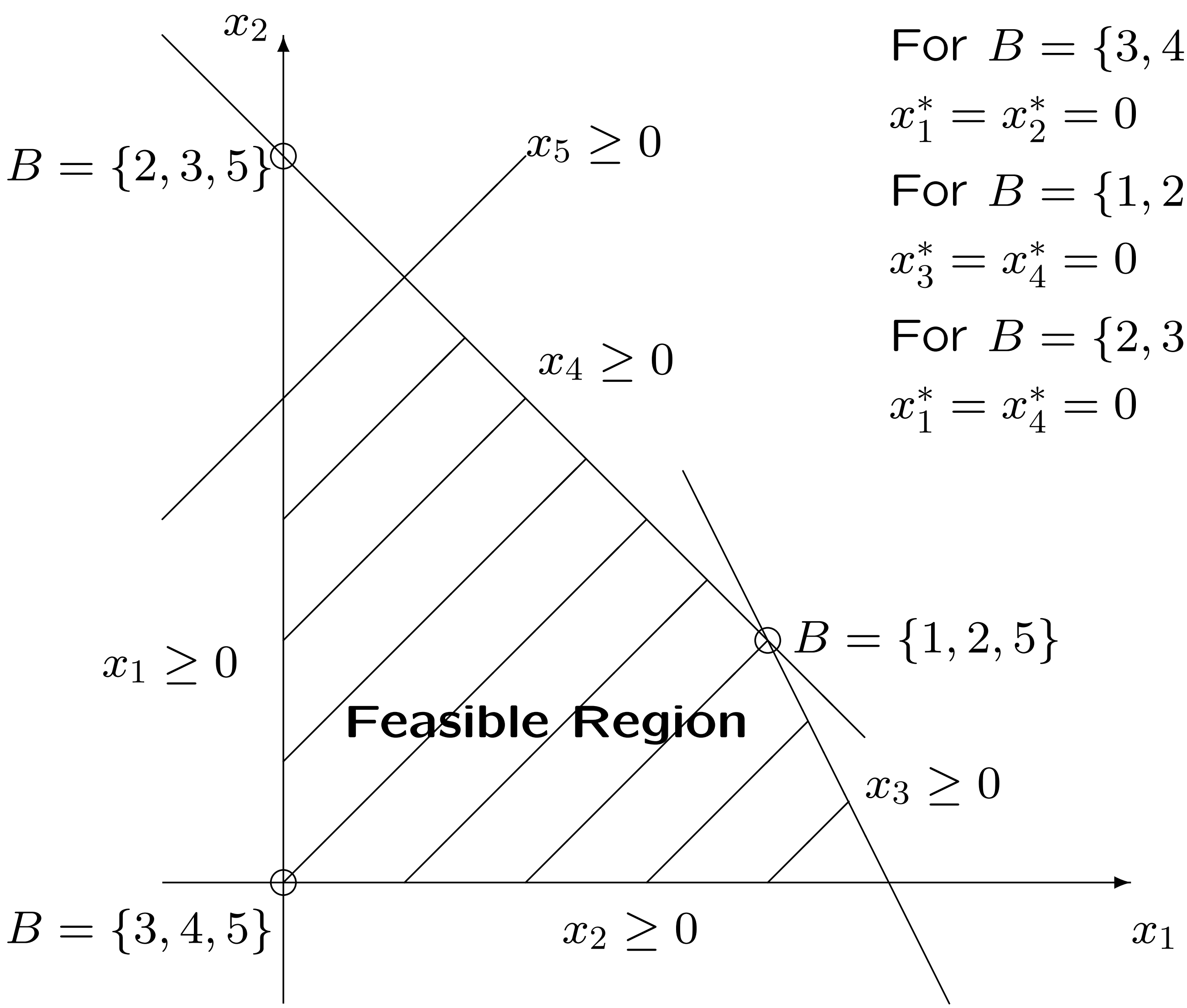
(Defn) **Degenerate basic solution**
A basic solution than has $< m$ non-zeros.
I.e., $x_i^* = 0$ for some $i \in B$.

We shall see a lot of degeneracy in
"Chapter 8: Degeneracy and Finite Termination"

Geometry of basic solutions

Consider the feasible region of the orange factory problem.

$$
\begin{array}{rcrcrcrcrcr}
2x_1 & + & x_2 & + & x_3 & & & & & = & 10 \\
x_1 & + & x_2 & & & + & x_4 & & & = & 6 \\
-x_1 & + & x_2 & & & & & + & x_5 & = & 4 \\
x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq & 0
\end{array}
$$



For $B = \{3,4,5\}$,
$x_1^* = x_2^* = 0$

For $B = \{1,2,5\}$,
$x_3^* = x_4^* = 0$

For $B = \{2,3,5\}$,
$x_1^* = x_4^* = 0$

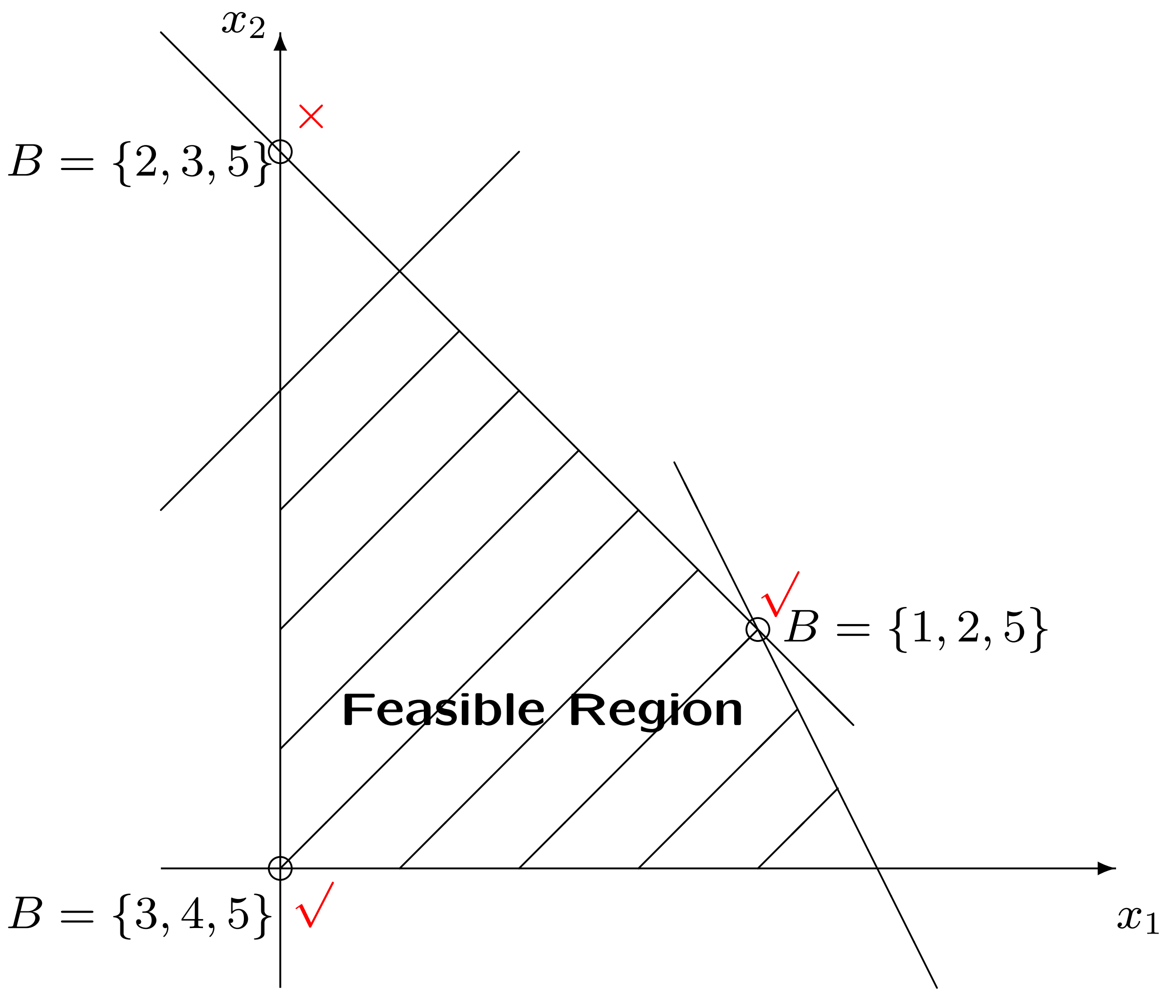
The basic solutions are intersections of two lines.

Basic Feasible Solutions

(Defn) **Basic feasible solution (bfs)**

Bfs of $\{Ax = b,\ x \geq 0\}$ is a basic solution of $Ax = b$ that is also non-negative.

Example



Basic solutions determined by $B = \{3, 4, 5\}$ and $B = \{1, 2, 5\}$ are basic feasible solutions.

Basic solution determined by $B = \{2, 3, 5\}$ is NOT a bfs.

Each bfs corresponds to a "corner point".

Convex sets

Observation 1: the feasible region "concaves out".

Mathematically, we say the feasible region is a convex set.

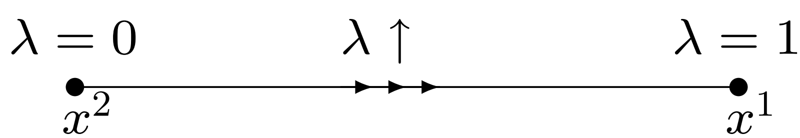
(Defn) **Convex sets (Pg 62)**
A set C in $\mathbf{R}^n$ is convex if
for every $x^1, x^2 \in C$ and every λ with $0 \leq \lambda \leq 1$,

$$\lambda x^1 + (1 - \lambda) x^2 \in C.$$

What does it mean?

$$\lambda x^1 + (1 - \lambda) x^2 = x^2 + \lambda(x^1 - x^2)$$

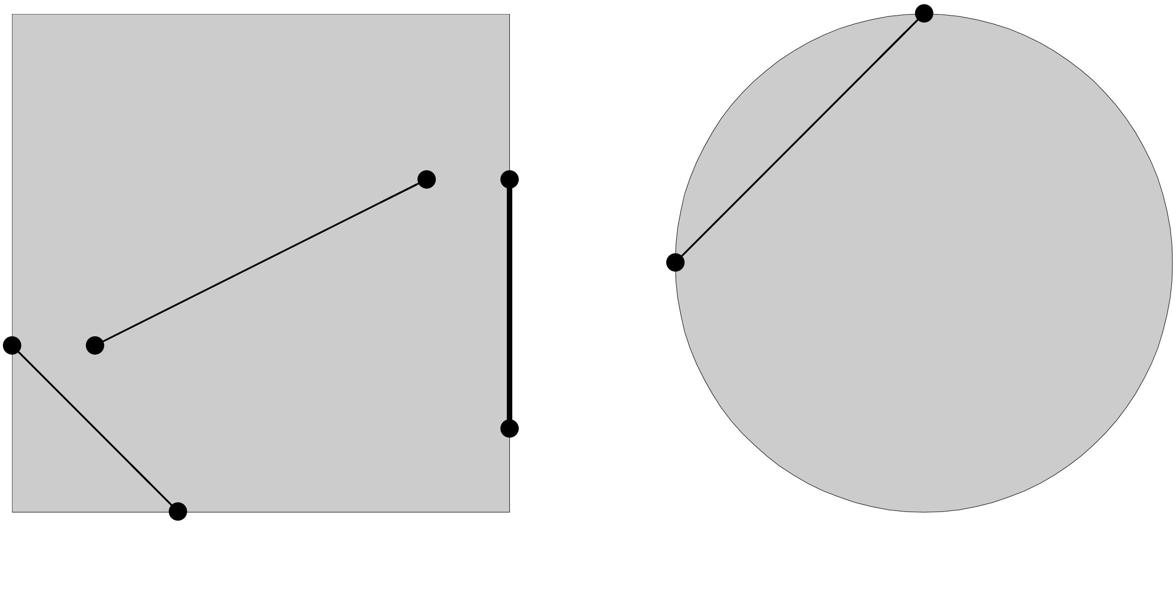
When $\lambda = 0$, we get x^2.
As λ increases from 0, we move in the direction $x^1 - x^2$.
When $\lambda = 1$, we get x^1.



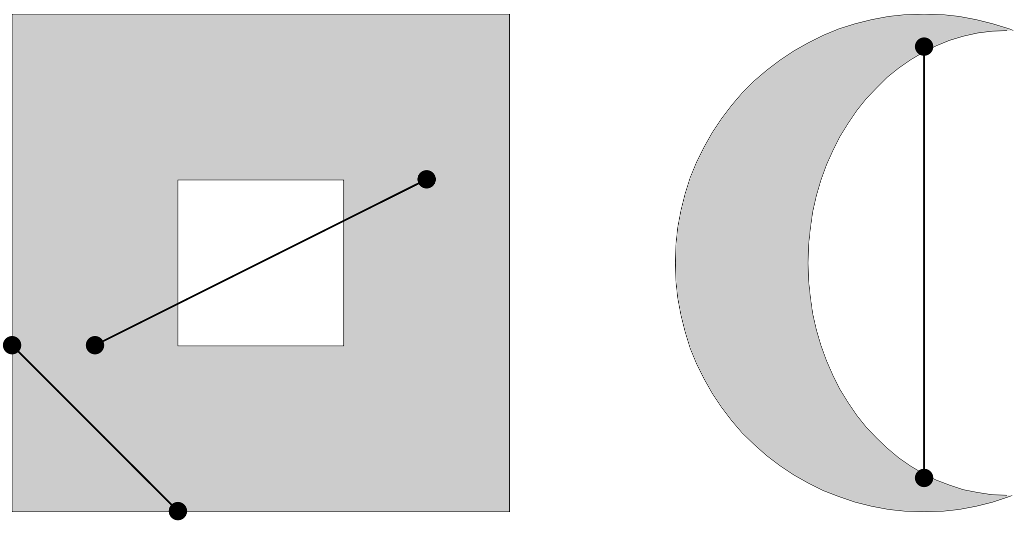
Thus $\lambda x^1 + (1 - \lambda) x^2$ represents vectors lying between x^1 and x^2 (including both x^1 and x^2).

So the definition of a convex set C reads
"for any two vectors in C, every vector lying between them is in C".

Examples of convex sets



Examples of non-convex sets



Proposition 5.2 (Pg 62) If F is the feasible region of a linear programming problem, then F is a convex set.

Proof: Recall definition of a convex set C:
"for any two vectors in C, every vector lying between them is in C".

We need to show that if we pick any two vectors in F, then every vector between them also lie in F.

Pick <u>any</u> x^1 and x^2 in F.
Pick <u>any</u> $0 \leq \lambda \leq 1$ and consider $x^3 = \lambda x^1 + (1-\lambda)x^2$.
We want to show that x^3 is in F.

F is described by a set of linear equations and inequalities.
We may assume that all inequalities are of the flavour "$\leq$".

For each equation $a^T x = b$:

$$\begin{aligned} a^T x^3 &= a^T(\lambda x^1 + (1-\lambda)x^2) \\ &= \lambda a^T x^1 + (1-\lambda)a^T x^2 \\ &= \lambda b + (1-\lambda)b = b \end{aligned}$$

For each inequality $a^T x \leq b$:

$$\begin{aligned} a^T x^3 &= a^T(\lambda x^1 + (1-\lambda)x^2) \\ &= \lambda a^T x^1 + (1-\lambda)a^T x^2 \\ &\leq \lambda b + (1-\lambda)b = b \end{aligned}$$

x^3 satisfies all equations and inequalities $\implies$ $x^3 \in F$. ■

Extreme point

Observation 2: a "corner point" in a convex set is a place where we cannot "fit" a straight line with non-zero length.

Mathematically, a "corner point" is called an extreme point.

(Defn) **Extreme point**

x^0 is an extreme point of a convex set C if

- x^0 lies in C, and
- there does not exist $x^1, x^2 \in C$ with $x^1 \neq x^2$ and $0 < \lambda < 1$ such that $x^0 = \lambda x^1 + (1-\lambda)x^2$.

What does it mean?

We know that $\lambda x^1 + (1-\lambda)x^2$ is a vector between x^1 and x^2 for all $0 \leq \lambda \leq 1$.

By excluding $\lambda = 0$ and $\lambda = 1$, we excludes both x^1 and x^2;

except when $x^1 = x^2$, in which case

$$\lambda x^1 + (1-\lambda)x^2 = x^1 = x^2$$

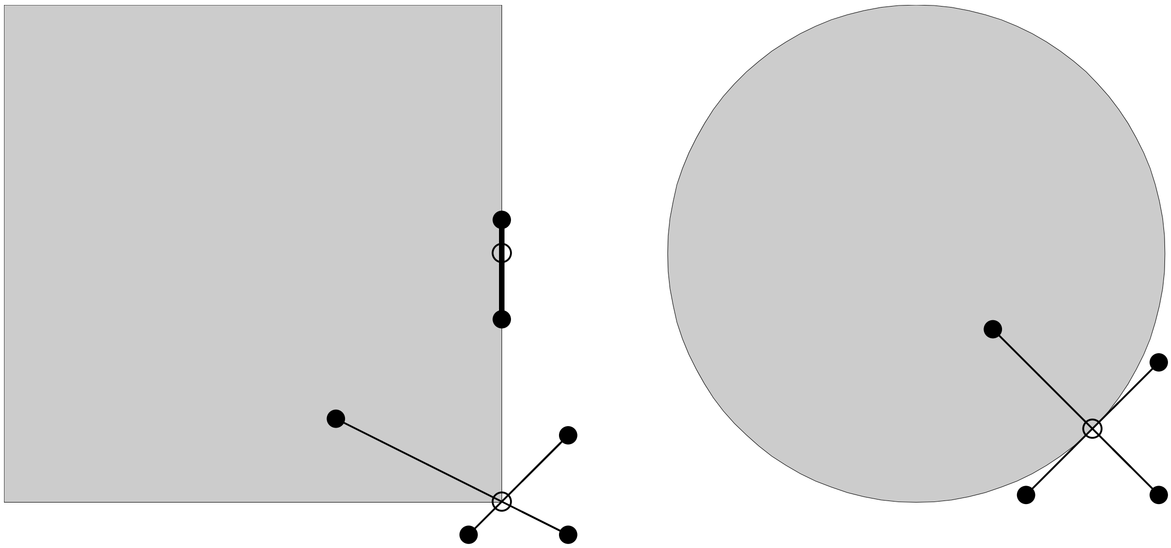
no matter what λ is.

So we further exclude the case $x^1 = x^2$.

The definition of an extreme point x^0 reads

"x^0 is in C and x^0 does not lie strictly between two vectors in C."

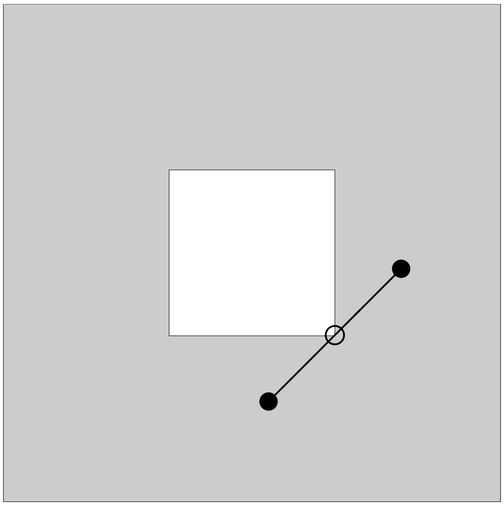
Examples of extreme points



The square has 4 extreme points.

The circle has infinitely many extreme points.

Extreme points for non-convex sets?



The inside "corner point" in this non-convex set does not match the definition of an extreme point.

BUT we only deal with convex sets in this course.