CO350 Linear Programming Chapter 4: Introduction to Duality

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Recap

LP in SEF:

$$(P) \quad \begin{array}{lrcl} \text{maximize} & c^T x & & \\ \text{subject to} & Ax & = & b \\ & x & \geq & 0 \end{array}$$

Dual LP:

$$(D) \quad \begin{array}{lrcl} \text{minimize} & b^T y & & \\ \text{subject to} & A^T y & \geq & c \end{array}$$

Complementary Slackness (CS) Condition

$$x_j^* = 0 \text{ or } \sum_{i=1}^{m} a_{ij} y_i^* = c_j \text{ (or both) for each } j$$

A more useful form:

$$x_j^* \neq 0 \implies \sum_{i=1}^{m} a_{ij} y_i^* = c_j \text{ for each } j$$

Theorem 4.7 (CS Theorem) Suppose x^* feasible for (P) and y^* feasible for (D).

x^* optimal for (P) and y^* optimal for (D)

$\iff$ CS condition holds for x^*, y^*.

Theorem 4.8 (CS Theorem restated) Suppose x^* is feasible for (P).

x^* optimal for (P)

$\iff$ there exists y^* feasible for (D) such that CS condition holds for x^*, y^*.

Complementary Slackness for Other Forms

CS condition for general LP (pg 47)

$$x_j^* = 0 \text{ or } \sum_{i=1}^{m} a_{ij} y_i^* = c_j \text{ for each } j$$

AND

$$y_i^* = 0 \text{ or } \sum_{j=1}^{m} a_{ij} x_j^* = b_i \text{ for each } i$$

Interpretation for SEF

In SEF, we have $Ax = b$ as constraints.

For any feasible x^*, we always have $\sum_{j=1}^{m} a_{ij} x_j^* = b_i$.

Therefore, the above CS condition reduces to

$$x_j^* = 0 \text{ or } \sum_{i=1}^{m} a_{ij} y_i^* = c_j \text{ for each } j$$

Similarly,

x_j is a free variable

$\Longrightarrow \quad \sum_{i=1}^{m} a_{ij} y_i^* = c_j$ is a constraint for dual LP

$\Longrightarrow \quad x_j^* = 0$ or $\sum_{i=1}^{m} a_{ij} y_i^* = c_j$ is redundant

Theorem 4.9 [Important]

Suppose x^* feasible for an LP and y^* feasible for dual LP.

$$\begin{aligned} & x^* \text{ and } y^* \text{ optimal for their resp. LPs} \\ \iff \quad & \text{CS condition holds for } x^*, y^* \end{aligned}$$

Proof for SIF: (Proof for general form is similar)

> Important:
> Know how to prove for LP problems in general form.

$$(P) \quad \begin{array}{lrcl} \max & c^T x & & \\ \text{s.t.} & Ax & \leq & b \\ & x & \geq & 0 \end{array} \quad \overset{\text{dual}}{\longleftrightarrow} \quad (D) \quad \begin{array}{lrcl} \min & b^T y & & \\ \text{s.t.} & A^T y & \geq & c \\ & y & \geq & 0 \end{array}$$

$\updownarrow$ equiv. $\qquad\qquad$ $\updownarrow$ same!

$$(\widehat{P}) \quad \begin{array}{lrcccl} \max & c^T x & & & & \\ \text{s.t.} & Ax & + & s & = & b \\ & x & , & s & \geq & 0 \end{array} \quad \overset{\text{dual}}{\longleftrightarrow} \quad (\widehat{D}) \quad \begin{array}{lrcl} \min & b^T y & & \\ \text{s.t.} & A^T y & \geq & c \\ & y & \geq & 0 \end{array}$$

$$x^* \text{ is } \begin{Bmatrix} \text{feasible} \\ \text{optimal} \end{Bmatrix} \text{ for } (P)$$

$$\Longrightarrow \begin{bmatrix} x^* \\ s^* \end{bmatrix} = \begin{bmatrix} x^* \\ b - Ax^* \end{bmatrix} \text{ is } \begin{Bmatrix} \text{feasible} \\ \text{optimal} \end{Bmatrix} \text{ for } (\widehat{P}).$$

Proof for SIF (cont'd):

$$(\widehat{P}) \quad \begin{array}{lrcrcl} \max & c^T x & & & & \\ \text{s.t.} & Ax & + & s & = & b \\ & x & , & s & \geq & 0 \end{array} \quad \overset{\text{dual}}{\longleftrightarrow} \quad (\widehat{D}) \quad \begin{array}{lrcl} \min & b^T y & & \\ \text{s.t.} & A^T y & \geq & c \\ & y & \geq & 0 \end{array}$$

x^* and y^* feasible for (P) and (D) resp.

$\Longrightarrow \begin{bmatrix} x^* \\ b - Ax^* \end{bmatrix}$ and y^* feasible for $(\widehat{P})$ and $(\widehat{D})$ resp.

x^* and y^* optimal for (P) and (D) resp.

$\Longleftrightarrow \begin{bmatrix} x^* \\ b - Ax^* \end{bmatrix}$ and y^* optimal for $(\widehat{P})$ and $(\widehat{D})$ resp.

Apply Theorem 4.7 (CS theorem) on $(\widehat{P})$ and $(\widehat{D})$:

$\begin{bmatrix} x^* \\ s^* \end{bmatrix} = \begin{bmatrix} x^* \\ b - Ax^* \end{bmatrix}$ and y^* optimal for $(\widehat{P})$ and $(\widehat{D})$ resp.

$\Longleftrightarrow$ $x_j^* = 0$ or $\sum_{i=1}^{m} a_{ij} y_i^* = c_j$ for each j

AND $(b - Ax^*)_i = 0$ or $y_i^* = 0$ for each i

$\Longleftrightarrow$ $x_j^* = 0$ or $\sum_{i=1}^{m} a_{ij} y_i^* = c_j$ for each j

AND $\sum_{j=1}^{m} a_{ij} x_j^* = b_i$ or $y_i^* = 0$ for each i $\square$

Application of CS Thm for other forms (Example)

(Read pg 48 for additional example)

$$
\begin{array}{lrcrcrcr}
\text{maximize} & x_1 & + & 3x_2 & + & x_3 & & \\
\text{subject to} & -2x_1 & + & 4x_2 & + & x_3 & = & -2 \\
 & 5x_1 & - & 5x_2 & - & x_3 & \le & 10 \\
 & x_1 & , & x_2 & , & x_3 & \ge & 0
\end{array}
\qquad (P')
$$

Question: Check optimality of each solution.

(i) $x^1 = [2, 0, 2]^T$ (ii) $x^2 = [\frac{8}{3}, 0, \frac{10}{3}]^T$ (iii) $x^3 = [\frac{14}{5}, \frac{2}{5}, 2]^T$

General approach:

1. Check x^* feasible for (P).
2. Write down equations for y^* from CS condition.
3. Try to solve for y^*.
4. Check y^* feasible for dual.

Dual problem:

$$
\begin{array}{lrcrcr}
\text{minimize} & -2y_1 & + & 10y_2 & & \\
\text{subject to} & -2y_1 & + & 5y_2 & \ge & 1 \\
 & 4y_1 & - & 5y_2 & \ge & 3 \\
 & y_1 & - & y_2 & \ge & 1 \\
 & & & y_2 & \ge & 0
\end{array}
\qquad (D')
$$

Application of CS Thm for other forms (Example)

$$
\begin{array}{lrcrcrcr}
\text{maximize} & x_1 & + & 3x_2 & + & x_3 & & \\
\text{subject to} & -2x_1 & + & 4x_2 & + & x_3 & = & -2 \\
 & 5x_1 & - & 5x_2 & - & x_3 & \leq & 10 \\
 & x_1 & , & x_2 & , & x_3 & \geq & 0
\end{array}
\qquad (P')
$$

Question: Check optimality of each solution.

(i) $x^1 = [2, 0, 2]^T$ (ii) $x^2 = [\frac{8}{3}, 0, \frac{10}{3}]^T$ (iii) $x^3 = [\frac{14}{5}, \frac{2}{5}, 2]^T$

Dual problem:

$$
\begin{array}{lrcrcr}
\text{minimize} & -2y_1 & + & 10y_2 & & \\
\text{subject to} & -2y_1 & + & 5y_2 & \geq & 1 \\
 & 4y_1 & - & 5y_2 & \geq & 3 \\
 & y_1 & - & y_2 & \geq & 1 \\
 & & & y_2 & \geq & 0
\end{array}
\qquad (D')
$$

CS condition: $\left[\begin{array}{l} x_j^* = 0 \text{ or } (A^T y^* - c)_j = 0 \text{ for each } j \\ y_i^* = 0 \text{ or } (Ax^* - b)_i = 0 \text{ for each } i \end{array} \right]$

$$
\begin{array}{lcrcl}
x_1^* = 0 & \text{or} & -2y_1^* + 5y_2^* & = & 1 \\
x_2^* = 0 & \text{or} & 4y_1^* - 5y_2^* & = & 3 \\
x_3^* = 0 & \text{or} & y_1^* - y_2^* & = & 1 \\
y_2^* = 0 & \text{or} & 5x_1^* - 5x_2^* - x_3 & = & 10
\end{array}
$$

Application of CS Thm for other forms (Example)

Primal constraints:

$$\begin{array}{rcrcrcrl} -2x_1 & + & 4x_2 & + & x_3 & = & -2 & \text{— (P1)} \\ 5x_1 & - & 5x_2 & - & x_3 & \leq & 10 & \text{— (P2)} \\ x_1 & , & x_2 & , & x_3 & \geq & 0 & \text{— (P3)} \end{array}$$

Dual constraints:

$$\begin{array}{rcrcrl} -2y_1 & + & 5y_2 & \geq & 1 & \text{— (D1)} \\ 4y_1 & - & 5y_2 & \geq & 3 & \text{— (D2)} \\ y_1 & - & y_2 & \geq & 1 & \text{— (D3)} \\ & & y_2 & \geq & 0 & \text{— (D4)} \end{array}$$

CS condition:

$$\begin{array}{rcrcrl} x_1^* \neq & 0 & \Longrightarrow & -2y_1^* + 5y_2^* & = 1 & \text{— (CS1)} \\ x_2^* \neq & 0 & \Longrightarrow & 4y_1^* - 5y_2^* & = 3 & \text{— (CS2)} \\ x_3^* \neq & 0 & \Longrightarrow & y_1^* - y_2^* & = 1 & \text{— (CS3)} \\ 5x_1^* - 5x_2^* - x_3 \neq & 10 & \Longrightarrow & y_2^* & = 0 & \text{— (CS4)} \end{array}$$

Question: Check optimality of each solution.

(i) $x^1 = [2, 0, 2]^T$ (ii) $x^2 = [\frac{8}{3}, 0, \frac{10}{3}]^T$ (iii) $x^3 = [\frac{14}{5}, \frac{2}{5}, 2]^T$

1. $x^1 = [2, 0, 2]^T$ satisfies $(P1) - (P3)$, so it is feasible.
2. From CS condition,

$$\begin{aligned} -2y_1^* + 5y_2^* &= 1 \quad \text{— (CS1)} \\ y_1^* - y_2^* &= 1 \quad \text{— (CS3)} \\ y_2^* &= 0 \quad \text{— (CS4)} \end{aligned}$$

3. The above system has no solution.

Conclusion: x^1 is not optimal.

Question: Check optimality of each solution.

(i) $x^1 = [2, 0, 2]^T$ (ii) $x^2 = [\frac{8}{3}, 0, \frac{10}{3}]^T$ (iii) $x^3 = [\frac{14}{5}, \frac{2}{5}, 2]^T$

1. $x^2 = [\frac{8}{3}, 0, \frac{10}{3}]^T$ satisfies $(P1) - (P3)$, so it is feasible.
2. From CS condition,

$$\begin{array}{rcrcl} -2y_1^* & + & 5y_2^* & = & 1 \quad \text{— (CS1)} \\ y_1^* & - & y_2^* & = & 1 \quad \text{— (CS3)} \end{array}$$

3. The above system has unique solution $y^* = [2, 1]^T$.
4. $y^* = [2, 1]^T$ satisfies $(D1) - (D4)$, so it is feasible.

Conclusion: x^2 is optimal.

Question: Check optimality of each solution.

(i) $x^1 = [2, 0, 2]^T$ (ii) $x^2 = [\frac{8}{3}, 0, \frac{10}{3}]^T$ <u>(iii)</u> $x^3 = [\frac{14}{5}, \frac{2}{5}, 2]^T$

1. $x^3 = [\frac{14}{5}, \frac{2}{5}, 2]^T$ satisfies $(P1) - (P3)$, so it is feasible.
2. From CS condition,

$$\begin{array}{rcrcl} -2y_1^* & + & 5y_2^* & = & 1 \text{ — (CS1)} \\ 4y_1^* & - & 5y_2^* & = & 3 \text{ — (CS2)} \\ y_1^* & - & y_2^* & = & 1 \text{ — (CS3)} \end{array}$$

3. The above system has unique solution $y^* = [2, 1]^T$.
4. $y^* = [2, 1]^T$ satisfies $(D1) - (D4)$, so it is feasible.

Conclusion: x^3 is optimal.