Introduction

LP problems takes many form — three types of constraints and two types of objectives.

Instead of developing theories and methods for all kinds of LP, we focus on TWO simple forms of LP problems

- Standard Inequality Form (SIF) and
- Standard Equality Form (SEF).

We shall show that all LP problems is "equivalent" to one in SIF and one in SEF.

Definitions

Standard Inequality Form (SIF)

$$\begin{array}{lrcll} \text{maximize} & c^T x & & & \\ \text{subject to} & Ax & \leq & b & \\ & x & \geq & 0 & \Longleftarrow \text{non-negativity constraints} \end{array}$$

1. It is a maximization problem.
2. All variables are required to be non-negative. These constraints are called non-negativity constraints.
3. All other constraints are linear inequalities of the type "$\leq$".

Example

The factory production problem is in SIF.

$$\begin{array}{lrcll} \text{maximize} & \sum_{j=1}^n c_j x_j & & & \\ \text{subject to} & \sum_{j=1}^n a_{ij} x_j & \leq & b_i & (i = 1, 2, \ldots, m) \\ & x_j & \geq & 0 & (j = 1, 2, \ldots, n) \end{array}$$

Standard Equality Form (SEF)

$$\begin{array}{lrcl} \text{maximize} & c^T x & & \\ \text{subject to} & Ax & = & b \\ & x & \geq & 0 \end{array}$$

1. It is a maximization problem.
2. All variables are required to be non-negative.
3. All other constraints are linear equations.

Example

A reformulation of the transportation problem is in SEF. (Compare with LP on page 12.)

$$\begin{array}{lrcll} \text{maximize} & -\sum_{i=1}^{p} \sum_{j=1}^{q} c_{ij} x_{ij} & & & \\ \text{subject to} & \sum_{j=1}^{q} x_{ij} & = & s_i & (i = 1, 2, \ldots, p) \\ & \sum_{i=1}^{p} x_{ij} & = & t_i & (j = 1, 2, \ldots, q) \\ & x_{ij} & \geq & 0 & \left(\begin{array}{l} i = 1, 2, \ldots, p, \\ j = 1, 2, \ldots, q \end{array} \right) \end{array}$$

Example

The problem of an overdetermined system of linear equations is not in SIF nor is it in SEF.

$$\begin{array}{lrcll}
\text{minimize} & \displaystyle\sum_{i=1}^{m} y_i & & & \\
\text{subject to} & \displaystyle\sum_{j=1}^{n} a_{ij}x_j - b_i - y_i & \leq & 0 & (i = 1, 2, \ldots, m) \\
& \displaystyle\sum_{j=1}^{n} a_{ij}x_j - b_i + y_i & \geq & 0 & (i = 1, 2, \ldots, m)
\end{array}$$

Fortunately, this LP problem — and all LP problems for that matter — can be **transformed** into an "equivalent" LP in standard form (i.e., SEF or SIF).

Transformations

We shall show that

> Every LP problem is "equivalent" to some LP problem in SIF (or SEF).

Constant term in objective functions

Constant term in objective can be removed.

> Can assume that the constant term is zero.

Does not change feasible region or optimal solutions.

Does change the optimal value (by the same constant).

Minimization Objective

Minimization problem: multiply objective function by -1.

> Minimize f is equivalent to maximize $-f$.

Does not change feasible region or optimal solutions.

Does change the optimal value (by factor of -1).

> **Note:** Some textbooks define standard forms as minimization problems. This same technique also changes maximization problems to minimization problems.
>
> (NOT USED IN THIS COURSE)

Equality Constraints

Each equality constraint can be replaced by two inequalities.

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

is equivalent to

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq b$$

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \geq b$$

Does not change feasible region, optimal solutions and the optimal value.

Inequality Constraints of type "$\geq$"

Multiply constraint by -1.

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \geq b$$

is equivalent to

$$-a_1x_1 - a_2x_2 - \cdots - a_nx_n \leq -b$$

Does not change feasible region, optimal solutions and the optimal value.

Note: Some textbooks define standard inequality form using "$\geq$" inequalities. This same technique also changes "$\leq$" constraints to "$\geq$" constraints.

(NOT USED IN THIS COURSE)

Unrestricted (a.k.a. free) Variables

(Defn) Variables that are not *explicitly* required to be non-negative.

Example: (Overdetermined system of equation)

(Compare with LP on page 14)

$$
\begin{array}{lrcll}
\text{minimize} & \displaystyle\sum_{i=1}^{m} y_i & & & \\
\text{subject to} & \displaystyle\sum_{j=1}^{n} a_{ij}x_j - y_i & \leq & b_i & (i = 1, 2, \ldots, m) \\
& \displaystyle -\sum_{j=1}^{n} a_{ij}x_j - y_i & \leq & -b_i & (i = 1, 2, \ldots, m)
\end{array}
$$

The variables x_j's and y_i's are free variables.

Equivalently,

$$
\begin{array}{lrcll}
\text{minimize} & \displaystyle\sum_{i=1}^{m} y_i & & & \\
\text{subject to} & \displaystyle\sum_{j=1}^{n} a_{ij}x_j - y_i & \leq & b_i & (i = 1, 2, \ldots, m) \\
& \displaystyle -\sum_{j=1}^{n} a_{ij}x_j - y_i & \leq & -b_i & (i = 1, 2, \ldots, m) \\
& y_i & \geq & 0 & (i = 1, 2, \ldots, m)
\end{array}
$$

The variables x_j's are free variables, but now y_i's are non-negative variables.

Unrestricted Variables (cont'd)

Note that every real number (zero, positive or negative) can be written as the difference of two non-negative numbers.
For example, $0 = 0 - 0$, $5 = 5 - 0$, $-3.4 = 0 - 3.4$.

Replace every occurrence of the free variable x_j by $u_j - v_j$, where $u_j, v_j \geq 0$.

Does not change optimal value.
Does change the feasible region and optimal solutions.

Given feasible solution of the old problem, there is an easy way to obtain a feasible solution of the transformed problem having the same objective value, and vice-versa.

So the two problems are "equivalent".

Unrestricted Variables (cont'd)

Example: (Overdetermined system of equation)

$$
\begin{array}{lrcll}
\text{minimize} & \displaystyle\sum_{i=1}^{m} y_i & & & \\
\text{subject to} & \displaystyle\sum_{j=1}^{n} a_{ij}x_j - y_i & \leq & b_i & (i = 1, 2, \ldots, m) \\
& \displaystyle -\sum_{j=1}^{n} a_{ij}x_j - y_i & \leq & -b_i & (i = 1, 2, \ldots, m) \\
& y_i & \geq & 0 & (i = 1, 2, \ldots, m)
\end{array}
$$

Replacing all x_j's by $u_j - v_j$ with $u_j, v_j \geq 0$:

$$
\begin{array}{lrcrcrcr}
\text{min} & & & & & \displaystyle\sum_{i=1}^{m} y_i & & \\
\text{s.t.} & \displaystyle\sum_{j=1}^{n} a_{ij}u_j & - & \displaystyle\sum_{j=1}^{n} a_{ij}v_j & - & y_i & \leq & b_i \\
& \displaystyle -\sum_{j=1}^{n} a_{ij}u_j & + & \displaystyle\sum_{j=1}^{n} a_{ij}v_j & - & y_i & \leq & -b_i \\
& u_j & , & v_j & , & y_i & \geq & 0
\end{array}
$$

$$
\begin{bmatrix} u^* \\ v^* \\ y^* \end{bmatrix} \text{ is } \left\{ \begin{array}{c} \text{feasible} \\ \text{optimal} \end{array} \right\} \iff \begin{bmatrix} u^* - v^* \\ y^* \end{bmatrix} \text{ is } \left\{ \begin{array}{c} \text{feasible} \\ \text{optimal} \end{array} \right\}.
$$

Lower-bounded Variables

$x_j \geq l_j$ if and only if $x_j - l_j \geq 0$

if and only if $x_j - l_j = x'_j$ with $x'_j \geq 0$

> Replace every occurrence of the lower-bounded variable x_j by $x'_j + l_j$. This changes (among other constraints) $x_j \geq l_j$ to $x'_j \geq 0$.

Upper-bounded Variables

$x_j \leq u_j$ if and only if $u_j - x_j = x'_j$ and $x'_j \geq 0$.

> Replace every occurrence of the upper-bounded variable x_j by $u_j - x'_j$. This changes (among other constraints) $x_j \leq u_j$ to $x'_j \geq 0$.

Does change the feasible region, optimal solutions and optimal value.

Given feasible solution of the old problem, there is an easy way to obtain a feasible solution of the transformed problem having the same objective value, and vice-versa.

So the two problems are "equivalent".

Note: Clearly, the constraint $-x_j \leq a$ is equivalent to the lower-bounded constraint $x_j \geq -a$. Similarly for $-x_j \geq b$.

Transformation into SIF

Using one or more of the above-mentioned transformations, we can transform any LP problem into an "equivalent" LP problem in SIF. (E.g., pg 30)

Example: Transform the LP into SIF. (NOT in notes)

$$
(P) \qquad \begin{array}{lrcrcl}
\text{minimize} & 2x_1 & - & x_2 & & \\
\text{subject to} & x_1 & + & 3x_2 & = & 3 \quad \text{—} \ (1) \\
 & x_1 & - & x_2 & \geq & 2 \quad \text{—} \ (2) \\
 & x_1 & & & \geq & 1 \quad \text{—} \ (3)
\end{array}
$$

1. Change minimization to maximization.

$$
\begin{array}{lrcrcl}
\text{maximize} & -2x_1 & + & x_2 & & \qquad \Longleftarrow \\
\text{subject to} & x_1 & + & 3x_2 & = & 3 \quad \text{—} \ (1) \\
 & x_1 & - & x_2 & \geq & 2 \quad \text{—} \ (2) \\
 & x_1 & & & \geq & 1 \quad \text{—} \ (3)
\end{array}
$$

2. Change equality constraint (1) to ineq. constraints.

$$
\begin{array}{lrcrcl}
\text{maximize} & -2x_1 & + & x_2 & & \\
\text{subject to} & x_1 & + & 3x_2 & \leq & 3 \quad \Longleftarrow (1a) \\
 & x_1 & + & 3x_2 & \geq & 3 \quad \Longleftarrow (1b) \\
 & x_1 & - & x_2 & \geq & 2 \quad \text{—} \ (2) \\
 & x_1 & & & \geq & 1 \quad \text{—} \ (3)
\end{array}
$$

Transformation into SIF (Example cont'd)

$$\begin{array}{lrcrcrl}
\text{maximize} & -2x_1 & + & x_2 & & & \\
\text{subject to} & x_1 & + & 3x_2 & \leq & 3 & — (1a) \\
 & x_1 & + & 3x_2 & \geq & 3 & — (1b) \\
 & x_1 & - & x_2 & \geq & 2 & — (2) \\
 & x_1 & & & \geq & 1 & — (3)
\end{array}$$

3. Change "$\geq$" constraints $(1a)$ and (2) to "$\leq$".

$$\begin{array}{lrcrcrl}
\text{maximize} & -2x_1 & + & x_2 & & & \\
\text{subject to} & x_1 & + & 3x_2 & \leq & 3 & — (1a) \\
 & -x_1 & - & 3x_2 & \leq & -3 & \Longleftarrow (1b') \\
 & -x_1 & + & x_2 & \leq & -2 & \Longleftarrow (2') \\
 & x_1 & & & \geq & 1 & — (3)
\end{array}$$

4. Change lower-bound (3) to non-negativity constraint. Replace all x_1 by $x_1' + 1$.

$$\begin{array}{lrcrcrl}
\text{maximize} & -2x_1' & + & x_2 & & \cancel{+2} & \Longleftarrow \\
\text{subject to} & x_1' & + & 3x_2 & \leq & 2 & \Longleftarrow \\
 & -x_1' & - & 3x_2 & \leq & -2 & \Longleftarrow \\
 & -x_1' & + & x_2 & \leq & -1 & \Longleftarrow \\
 & x_1' & & & \geq & 0 & \Longleftarrow
\end{array}$$

Transformation into SIF (Example cont'd)

$$
\begin{array}{lrcrcr}
\text{maximize} & -2x_1' & + & x_2 & & \cancel{+2} \\
\text{subject to} & x_1' & + & 3x_2 & \leq & 2 \\
 & -x_1' & - & 3x_2 & \leq & -2 \\
 & -x_1' & + & x_2 & \leq & -1 \\
 & x_1' & & & \geq & 0
\end{array}
$$

5. Replace free variable x_2 by $u_2 - v_2$ with $u_2, v_2 \geq 0$.

$$
(P') \quad
\begin{array}{lrcrcrcrl}
\text{maximize} & -2x_1' & + & u_2 & - & v_2 & & \cancel{+2} & \Longleftarrow \\
\text{subject to} & x_1' & + & 3u_2 & - & 3v_2 & \leq & 2 & \Longleftarrow \\
 & -x_1' & - & 3u_2 & + & 3v_2 & \leq & -2 & \Longleftarrow \\
 & -x_1' & + & u_2 & - & v_2 & \leq & -1 & \Longleftarrow \\
 & x_1' & , & u_2 & , & v_2 & \geq & 0 & \Longleftarrow
\end{array}
$$

Transformation into SIF (Example cont'd)

In summary, we start from

$$(P) \qquad \begin{array}{lrcrcrl} \text{minimize} & 2x_1 & - & x_2 & & & \\ \text{subject to} & x_1 & + & 3x_2 & = & 3 & \text{— (1)} \\ & x_1 & - & x_2 & \geq & 2 & \text{— (2)} \\ & x_1 & & & \geq & 1 & \text{— (3)} \end{array}$$

and use the transformations

1. Change minimization to maximization.
2. Change equality constraint to ineq. constraints.
3. Change "$\geq$" constraints to "$\leq$" constraints.
4. Change lower-bound variable $x_1 \geq 1$ to non-negative variable.
5. Replace free variable x_2 by $u_2 - v_2$ with $u_2, v_2 \geq 0$.

to get an equivalent LP problem (P') in SIF.

$$\begin{bmatrix} x_1^* \\ u_2^* \\ v_2^* \end{bmatrix} \text{ is } \left\{ \begin{array}{c} \text{feasible} \\ \text{optimal} \end{array} \right\} \iff \begin{bmatrix} x_1^* + 1 \\ u_2^* - v_2^* \end{bmatrix} \text{ is } \left\{ \begin{array}{c} \text{feasible} \\ \text{optimal} \end{array} \right\}.$$

v^* is opt. value of $(P') \iff -(v^* - 2)$ is opt. value of (P).