Beginning With an Example

The Orange Factory Problem

Factory produces two products:

Frozen concentrate and Orange juice.

Each unit of	requires	and produces
Frozen concentrate	2 units electricity 1 unit orange	$2 profit 1 unit water
Orange juice	1 unit electricity 1 unit orange 1 unit water	$3 profit

How much frozen concentrate and orange juice should we produce to maximize profit?

Daily available resources:

- 10 units electricity;
- 6 units oranges;
- 4 units water.

Decision variables

$\left\{ \begin{array}{c} x_1 \\ x_2 \end{array} \right\}$ denote daily production of $\left\{ \begin{array}{c} \text{frozen concentrate} \\ \text{orange juice} \end{array} \right\}$

In summary,

	Frozen concentrate	Orange juice	Resource limit
Decision variable	x_1	x_2	
Profit	2	3	
Electricity	2	1	10
Oranges	1	1	6
Water	-1	1	4

Mathematical model (Eq (1.1) on Pg 8)

$$\begin{array}{lrcrcr} \text{maximize} & 2x_1 & + & 3x_2 & & \\ \text{subject to} & & & & & \\ & 2x_1 & + & x_2 & \leq & 10 \\ & x_1 & + & x_2 & \leq & 6 \\ & -x_1 & + & x_2 & \leq & 4 \\ & x_1 & , & x_2 & \geq & 0 \end{array}$$

x_1 and x_2 need not be integers.

Solving the mathematical model (NOT in notes)

$$\begin{array}{llcrcl}
\text{maximize} & 2x_1 & + & 3x_2 & & \\
\text{subject to} & & & & & \\
& 2x_1 & + & x_2 & \leq & 10 \quad \text{— (1)} \\
& x_1 & + & x_2 & \leq & 6 \quad \text{— (2)} \\
& -x_1 & + & x_2 & \leq & 4 \quad \text{— (3)} \\
& x_1 & , & x_2 & \geq & 0
\end{array}$$

$(2) + (3) : 2x_2 \leq 10 \implies x_2 \leq 5$ — (4)

So, profit $= 2x_1 + 3x_2 = 2(x_1 + x_2) + x_2$.

From (2), $2(x_1 + x_2) \leq 2(6) = 12$.

From (4), $x_2 \leq 5$.

Thus, profit $\leq 12 + 5 = 17$.

$x_1 = 1$ and $x_2 = 5$ satisfy all inequalities and achieves a profit of 17.
i.e., $(x_1, x_2) = (1, 5)$ is an optimal solution.

General Production Problem

A factory makes n products from m resources. Each unit of product j requires a_{ij} units of resource i and makes a profit of c_j dollars. Each day, the factory has b_i units of resource i available.

> How much of each product should the factory make each day to maximize profit?

Decision variables: $x_j =$ daily production level of product j.

Objective: maximize profit $= \sum_{j=1}^{n} c_j x_j$.

Resource limit: total resource i used $= \sum_{j=1}^{n} a_{ij} x_j \leq b_i$.

Implicit restriction: $x_j \geq 0$.

Mathematical model

$$\begin{array}{lrcll} \text{maximize} & \sum_{j=1}^{n} c_j x_j & & & \\ \text{subject to} & \sum_{j=1}^{n} a_{ij} x_j & \leq & b_i & (i = 1, 2, \ldots, m) \\ & x_j & \geq & 0 & (j = 1, 2, \ldots, n) \end{array}$$

This is a linear programming problem or a linear program.

LP means linear programming or linear program.

Definition: LP Problem

Basic ingredients

Linear function:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

Linear equation:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

Linear inequality:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq b$$

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \geq b$$

NO "$<$" or "$>$"

Definition of LP problem: (Pg 9)
minimizing or maximizing a linear function subject to finite number of linear equalities and/or linear inequalities.

Examples of LP: See page 9 of notes.

Examples that are not LP problems (Pg 9)

$$\begin{array}{lrcrcl} \text{maximize} & x_1 & + & (x_2)^2 & & \\ \text{subject to} & x_1 & + & x_2 & = & 2 \\ & x_1 & & & \geq & 0 \end{array}$$

$$\begin{array}{lrcrcl} \text{maximize} & x_1 & + & x_2 & & \\ \text{subject to} & 2x_1 & + & x_2 & \leq & 6 \\ & e^{x_1} & & & \leq & 2 \end{array}$$

$$\begin{array}{lrcrcl} \text{maximize} & x_1 & + & x_2 & & \\ \text{subject to} & x_1 & + & 2x_2 & = & 3 \\ & x_1 & - & x_2 & \leq & 1 \\ & x_1 & & & > & 0 \end{array}$$