Section 6.2 Orthogonal Sets

A set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ in $\mathbf{R}^n$ is called an **orthogonal set** if $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ whenever $i \neq j$.

EXAMPLE: Is $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ an orthogonal set?

Solution: Label the vectors $\mathbf{u}_1, \mathbf{u}_2,$ and $\mathbf{u}_3$ respectively. Then

$\mathbf{u}_1 \cdot \mathbf{u}_2 =$

$\mathbf{u}_1 \cdot \mathbf{u}_3 =$

$\mathbf{u}_2 \cdot \mathbf{u}_3 =$

Therefore, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal set.

THEOREM 4

Suppose $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ is an orthogonal set of nonzero vectors in $\mathbf{R}^n$ and $W =$span$\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$. Then S is a linearly independent set and is therefore a basis for W.

Partial Proof: Suppose

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p = \mathbf{0}$$

$$(c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p) \cdot \qquad\qquad = \mathbf{0} \cdot$$

$$(c_1\mathbf{u}_1) \cdot \mathbf{u}_1 + (c_2\mathbf{u}_2) \cdot \mathbf{u}_1 + \cdots + (c_p\mathbf{u}_p) \cdot \mathbf{u}_1 = \mathbf{0}$$

$$c_1(\mathbf{u}_1 \cdot \mathbf{u}_1) + c_2(\mathbf{u}_2 \cdot \mathbf{u}_1) + \cdots + c_p(\mathbf{u}_p \cdot \mathbf{u}_1) = \mathbf{0}$$

$$c_1(\mathbf{u}_1 \cdot \mathbf{u}_1) = \mathbf{0}$$

Since $\mathbf{u}_1 \neq \mathbf{0}$, $\mathbf{u}_1 \cdot \mathbf{u}_1 > 0$ which means $c_1 =$ ____.

In a similar manner, $c_2, \ldots, c_p$ can be shown to by all 0. So S is a linearly independent set. ■

An **orthogonal basis** for a subspace W of $\mathbf{R}^n$ is a basis for W that is also an orthogonal set.

EXAMPLE: Suppose $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ is an orthogonal basis for a subspace W of $\mathbf{R}^n$ and suppose $\mathbf{y}$ is in W. Find $c_1, \ldots, c_p$ so that

$$\mathbf{y} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p.$$

Solution:

$$\mathbf{y} \cdot \qquad = (c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p) \cdot$$

$$\mathbf{y} \cdot \mathbf{u}_1 = (c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p) \cdot \mathbf{u}_1$$

$$\mathbf{y} \cdot \mathbf{u}_1 = c_1(\mathbf{u}_1 \cdot \mathbf{u}_1) + c_2(\mathbf{u}_2 \cdot \mathbf{u}_1) + \cdots + c_p(\mathbf{u}_p \cdot \mathbf{u}_1)$$

$$\mathbf{y} \cdot \mathbf{u}_1 = c_1(\mathbf{u}_1 \cdot \mathbf{u}_1)$$

$$c_1 = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}$$

Similarly, $c_2 =$, $c_3 =$, $\ldots,$ $c_p =$

THEOREM 5

Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ be an orthogonal basis for a subspace W of $\mathbf{R}^n$. Then each $\mathbf{y}$ in W has a unique representation as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p$. In fact, if

$$\mathbf{y} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p$$

then

$$c_j = \frac{\mathbf{y} \cdot \mathbf{u}_j}{\mathbf{u}_j \cdot \mathbf{u}_j} \qquad (j = 1, \ldots, p)$$

EXAMPLE: Express $\mathbf{y} = \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ as a linear combination of the orthogonal basis

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Solution:

$$\frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1} = \qquad \frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2} = \qquad \frac{\mathbf{y}\cdot\mathbf{u}_3}{\mathbf{u}_3\cdot\mathbf{u}_3} =$$

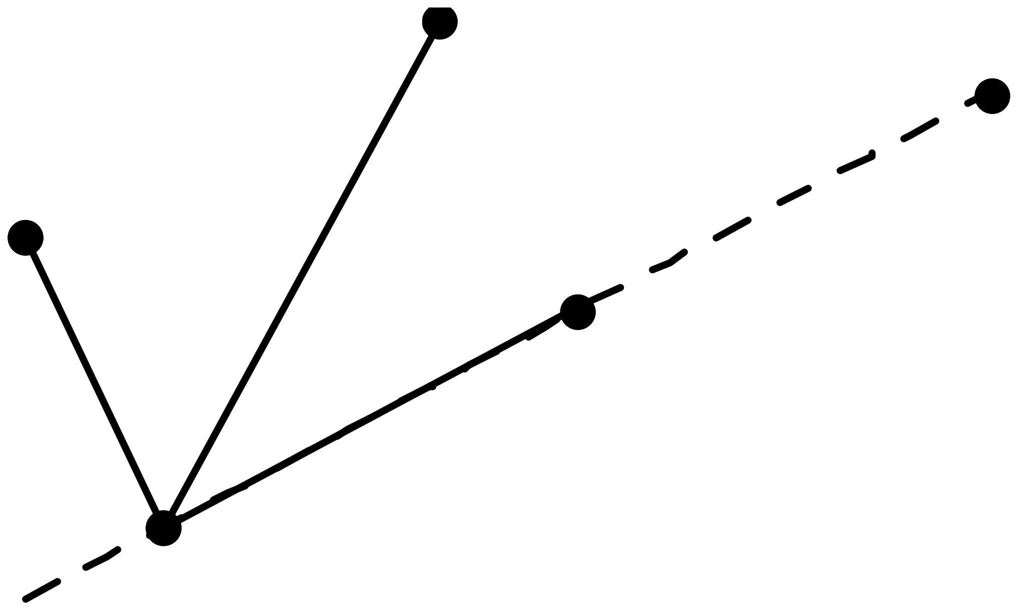
Hence

$$\mathbf{y} = \_\_\_\_\_\mathbf{u}_1 + \_\_\_\_\_\_\mathbf{u}_2 + \_\_\_\_\_\_\mathbf{u}_3$$

Orthogonal Projections

For a nonzero vector **u** in $\mathbf{R}^n$, suppose we want to write **y** in $\mathbf{R}^n$ as the the following

$$\mathbf{y} = (\text{multiple of } \mathbf{u}) + (\text{multiple a vector } \perp \text{ to } \mathbf{u})$$



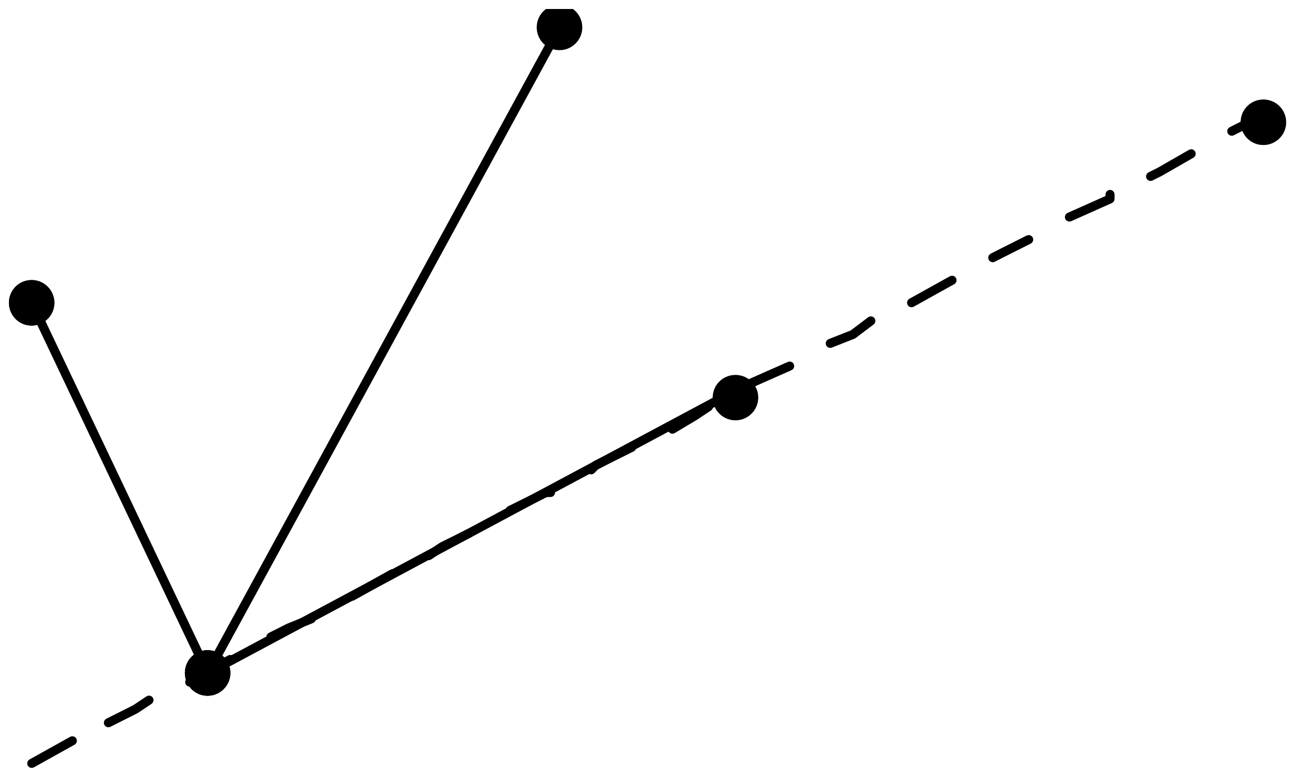
$$(\mathbf{y} - \alpha\mathbf{u}) \cdot \mathbf{u} = 0$$

$$\mathbf{y} \cdot \mathbf{u} - \alpha(\mathbf{u} \cdot \mathbf{u}) = 0 \qquad \Rightarrow \qquad \alpha =$$

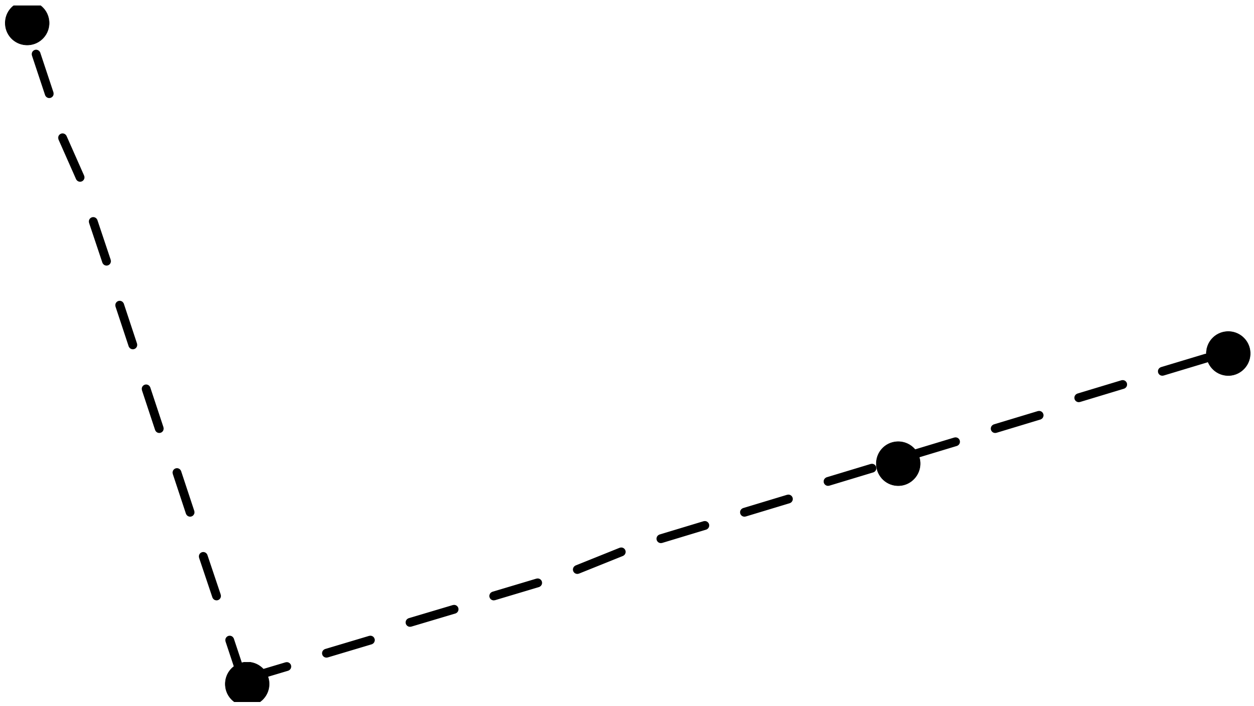
$$\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} \qquad \text{(orthogonal projection of y onto u)}$$

and

$$\mathbf{z} = \mathbf{y} - \frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} \qquad \text{(component of y orthogonal to u)}$$



EXAMPLE: Let $\mathbf{y} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Compute the distance from **y** to the line through **0** and **u**.



Solution:

$$\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} =$$

Distance from **y** to the line through **0** and **u** $=$ distance from $\hat{\mathbf{y}}$ to **y**

$$= \|\hat{\mathbf{y}} - \mathbf{y}\| =$$

Orthonormal Sets

A set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ in $\mathbf{R}^n$ is called an **orthonormal set** if it is an orthogonal set of unit vectors.

If $W =$span$\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$, then $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ is an orthonormal basis for W.

Recall that $\mathbf{v}$ is a unit vector if $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{\mathbf{v}^T\mathbf{v}} = 1$.

Suppose $U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix}$ where $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal set.

Then $U^TU = \begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \mathbf{u}_3^T \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$

$= \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$

It can be shown that $UU^T = I$ also. So $U^{-1} = U^T$ (such a matrix is called an **orthogonal matrix**).

THEOREM 6 An $m \times n$ matrix U has orthonormal columns if and only if $U^T U = I$.

THEOREM 7 Let U be an $m \times n$ matrix with orthonormal columns, and let **x** and **y** be in $\mathbf{R}^n$. Then

a. $\|U\mathbf{x}\| = \|\mathbf{x}\|$

b. $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$

c. $(U\mathbf{x}) \cdot (U\mathbf{y}) = 0$ if and only if $\mathbf{x} \cdot \mathbf{y} = 0$.

Proof of part b: $(U\mathbf{x}) \cdot (U\mathbf{y}) =$