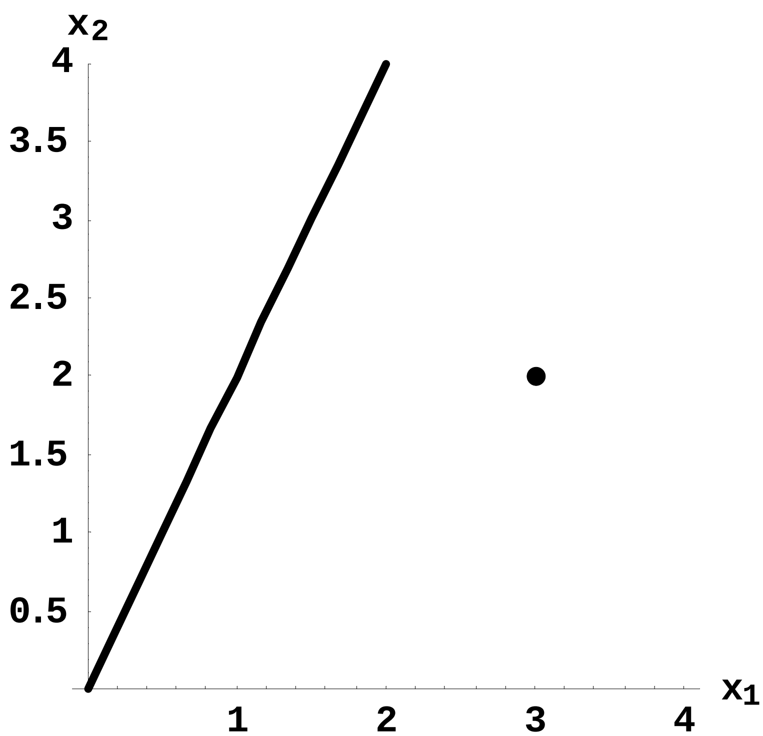
## 6.1 Inner Product, Length & Orthogonality

Not all linear systems have solutions.

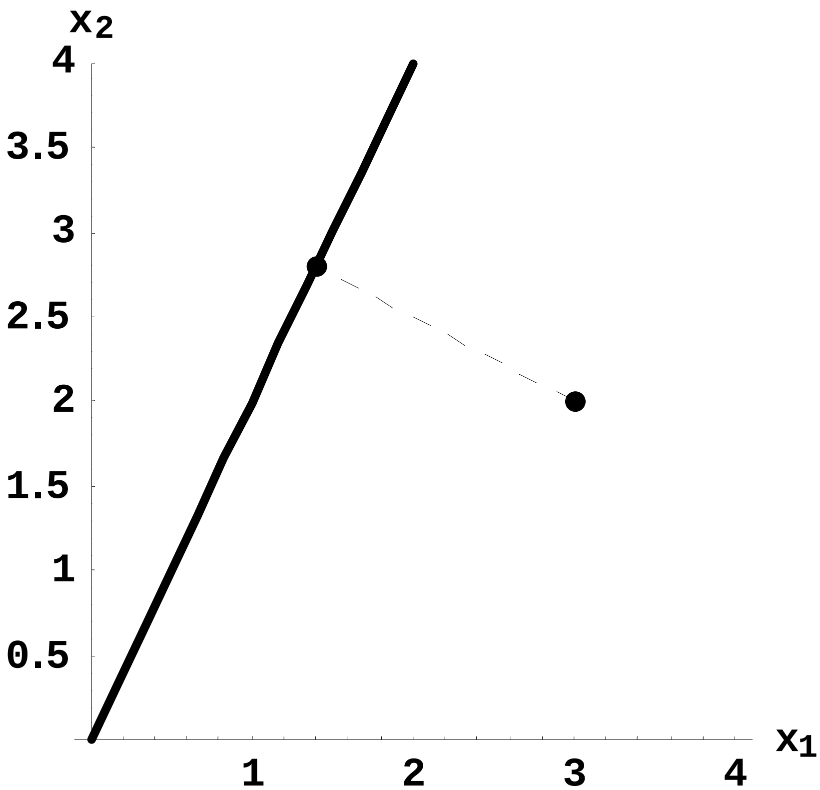
**EXAMPLE:** No solution to $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ exists.

Why?

$A\mathbf{x}$ is a point on the line spanned by $\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$ and **b** is not on the line. So $A\mathbf{x} \neq \mathbf{b}$ for all **x**.



Instead find $\hat{\mathbf{x}}$ so that $A\hat{\mathbf{x}}$ lies "closest" to $\mathbf{b}$.




Using information we will learn in this chapter, we will find that

$\hat{\mathbf{x}} = \begin{bmatrix} 1.4 \\ 0 \end{bmatrix}$, so that $A\hat{\mathbf{x}} = \begin{bmatrix} 1.4 \\ 2.8 \end{bmatrix}$.

Segment joining $A\hat{\mathbf{x}}$ and $\mathbf{b}$ is *perpendicular (*or *orthogonal)* to the set of solutions to $A\mathbf{x} = \mathbf{b}$.

Need to develop fundamental ideas of *length*, *orthogonality* and *orthogonal projections*.

## The Inner Product

**Inner product** or **dot product** of

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}:$$

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T\mathbf{v} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} =$$

$$u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

Note that

$$\mathbf{v} \cdot \mathbf{u} = v_1u_1 + v_2u_2 + \cdots + v_nu_n$$

$$= u_1v_1 + u_2v_2 + \cdots + u_nv_n = \mathbf{u} \cdot \mathbf{v}$$

**THEOREM 1**

Let $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbf{R}^n$, and let $c$ be any scalar. Then

a. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

b. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

c. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$

d. $\mathbf{u} \cdot \mathbf{u} \geq \mathbf{0}$, and $\mathbf{u} \cdot \mathbf{u} = \mathbf{0}$ if and only if $\mathbf{u} = \mathbf{0}$.

Combining parts b and c, one can show

$$\boxed{(c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p) \cdot \mathbf{w} = c_1(\mathbf{u}_1 \cdot \mathbf{w}) + \cdots + c_p(\mathbf{u}_p \cdot \mathbf{w})}$$

## Length of a Vector

For $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$, the **length** or **norm of v** is the nonnegative scalar $\|\mathbf{v}\|$ defined by

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \qquad \text{and} \qquad \|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v}.$$

For example, if $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$, then $\|\mathbf{v}\| = \sqrt{a^2 + b^2}$ (distance between **0** and **v**)

Picture:

For any scalar $c$,

$$\|c\mathbf{v}\| = |c|\,\|\mathbf{v}\|$$

## Distance in $\mathbf{R}^n$

The **distance between u and v** in $\mathbf{R}^n$:

$$\text{dist}(\mathbf{u},\mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|.$$

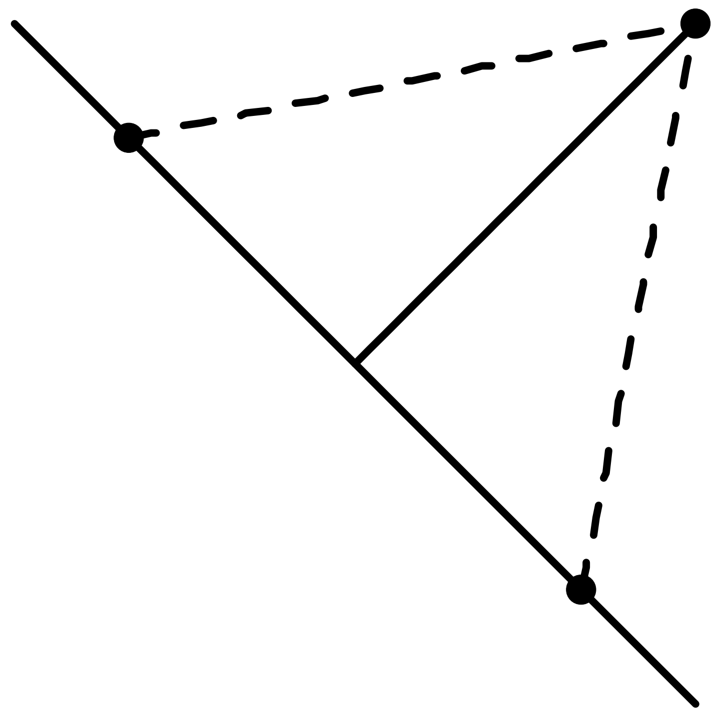
This agrees with the usual formulas for $\mathbf{R}^2$ and $\mathbf{R}^3$. Let $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$.

Then $\mathbf{u} - \mathbf{v} = (u_1 - v_1, u_2 - v_2)$ and

$$\text{dist}(\mathbf{u},\mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \|(u_1 - v_1, u_2 - v_2)\|$$

$$= \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2}$$

## Orthogonal Vectors



$$[\text{dist}(\mathbf{u},\mathbf{v})]^2 = \|\mathbf{u}-\mathbf{v}\|^2 = (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})$$

$$= (\mathbf{u})\cdot(\mathbf{u}-\mathbf{v}) + (-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) =$$

$$= \mathbf{u}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} + -\mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v}$$

$$= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\mathbf{u}\cdot\mathbf{v}$$

$$\Rightarrow \qquad [\text{dist}(\mathbf{u},\mathbf{v})]^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\mathbf{u}\cdot\mathbf{v}$$

Similarly,

$$[\text{dist}(\mathbf{u},-\mathbf{v})]^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\mathbf{u}\cdot\mathbf{v}$$

Since $[\text{dist}(\mathbf{u}, -\mathbf{v})]^2 = [\text{dist}(\mathbf{u}, \mathbf{v})]^2$, $\mathbf{u} \cdot \mathbf{v} =$ **____**.

Two vectors **u** and **v** are said to be **orthogonal** (to each other) if $\mathbf{u} \cdot \mathbf{v} = \mathbf{0}$.

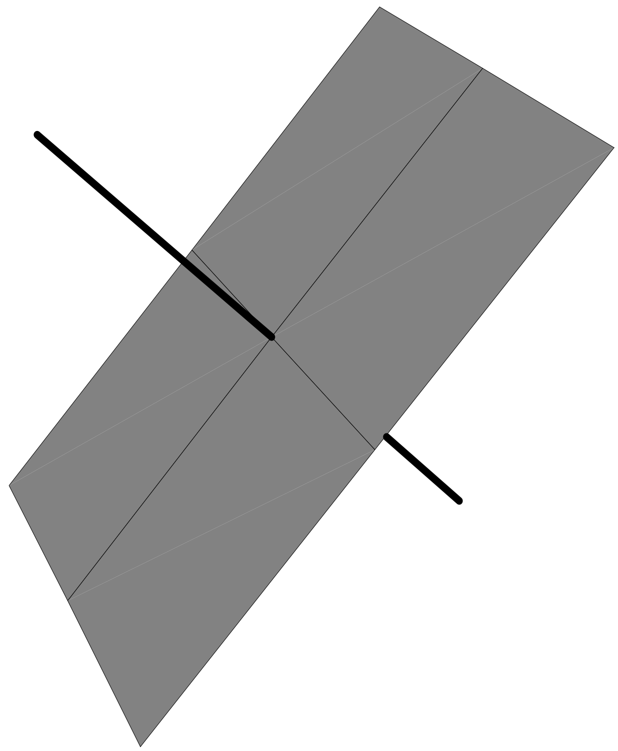
Also note that if **u** and **v** are orthogonal, then $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

**THEOREM 2 THE PYTHAGOREAN THEOREM**

Two vectors **u** and **v** are orthogonal if and only if $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

## Orthogonal Complements

If a vector **z** is orthogonal to every vector in a subspace $W$ of $\mathbf{R}^n$, then **z** is said to be **orthogonal to** $W$. The set of vectors **z** that are orthogonal to $W$ is called the **orthogonal complement** of $W$ and is denoted by $W^{\perp}$ (read as “W perp”).



# Row, Null and Columns Spaces

**THEOREM 3**

Let $A$ be an $m \times n$ matrix. Then the orthogonal complement of the row space of $A$ is the nullspace of $A$, and the orthogonal complement of the column space of $A$ is the nullspace of $A^T$:

$$(\text{Row } A)^{\perp} = \text{Nul } A, \qquad (\text{Col } A)^{\perp} = \text{Nul } A^T.$$

**Why? (See complete proof in the text)** Consider $A\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} * & * & \cdots & * \\ * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * \end{bmatrix} \begin{bmatrix} \star \\ \star \\ \vdots \\ \star \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Note that $A\mathbf{x} = \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{x} \\ \mathbf{r}_2 \cdot \mathbf{x} \\ \vdots \\ \mathbf{r}_m \cdot \mathbf{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ and so $\mathbf{x}$ is orthogonal to the row $A$ since $\mathbf{x}$ is orthogonal to $\mathbf{r}_1. \ldots, \mathbf{r}_m$.

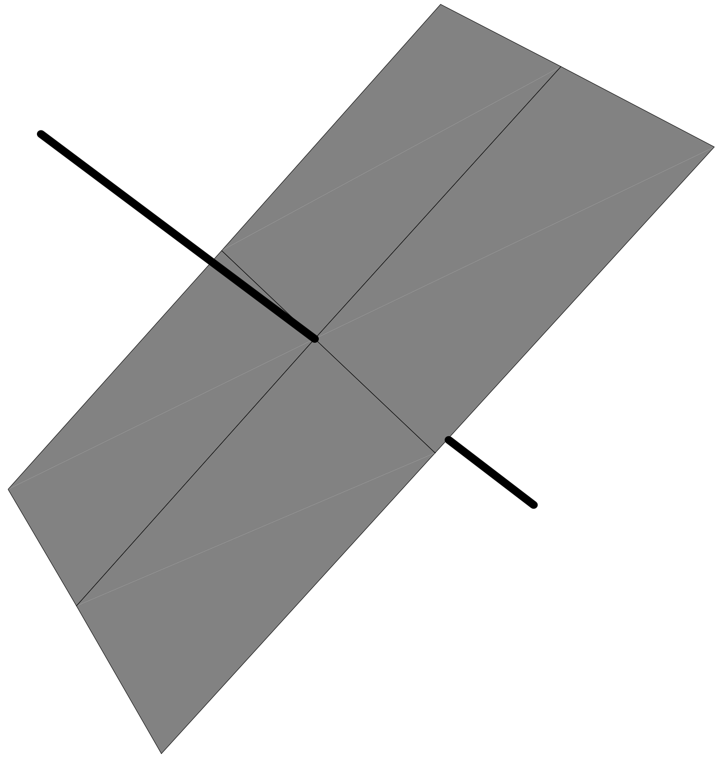
**EXAMPLE:** Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 2 \end{bmatrix}$.

Basis for Nul $A = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ and therefore Nul $A$ is a plane in $\mathbf{R}^3$.

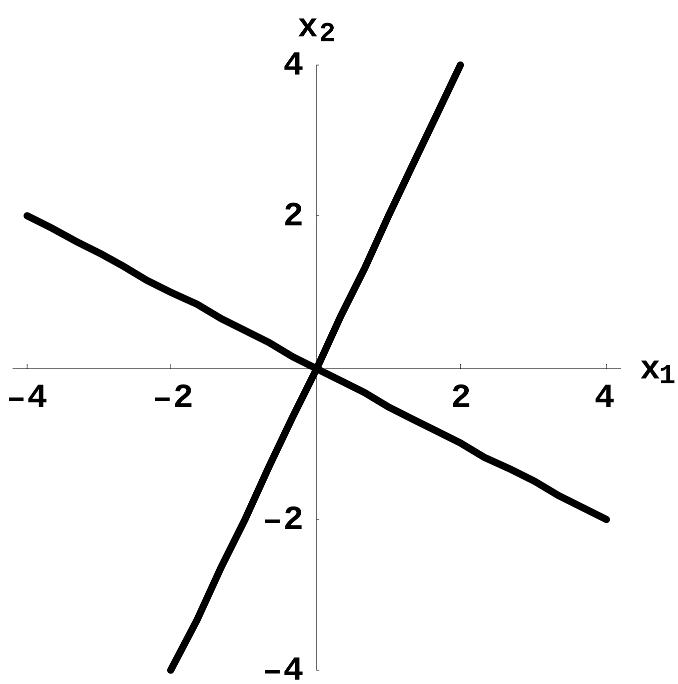
Basis for Row $A = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$ and therefore Row $A$ is a line in $\mathbf{R}^3$.

Basis for Col $A = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ and therefore Col $A$ is a line in $\mathbf{R}^2$.

Basis for Nul $A^T = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$ and therefore Nul $A^T$ is a line in $\mathbf{R}^2$.



Subspaces Nul $A$ and Row $A$




Subspaces Nul $A^T$ and Col $A$