2.3 Characterizations of Invertible Matrices

Theorem 8 (The Invertible Matrix Theorem)

Let A be a square $n \times n$ matrix. The the following statements are equivalent (i.e., for a given A, they are either all true or all false).

a. A is an invertible matrix.

b. A is row equivalent to I_n.

c. A has n pivot positions.

d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

e. The columns of A form a linearly independent set.

f. The linear transformation $\mathbf{x} \to A\mathbf{x}$ is one-to-one.

g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbf{R}^n$.

h. The columns of A span $\mathbf{R}^n$.

i. The linear transformation $\mathbf{x} \to A\mathbf{x}$ maps $\mathbf{R}^n$ onto $\mathbf{R}^n$.

j. There is an $n \times n$ matrix C such that $CA = I_n$.

k. There is an $n \times n$ matrix D such that $AD = I_n$.

l. A^T is an invertible matrix.

EXAMPLE: Use the Invertible Matrix Theorem to determine if A is invertible, where

$$A = \begin{bmatrix} 1 & -3 & 0 \\ -4 & 11 & 1 \\ 2 & 7 & 3 \end{bmatrix}.$$

Solution

$$A = \begin{bmatrix} 1 & -3 & 0 \\ -4 & 11 & 1 \\ 2 & 7 & 3 \end{bmatrix} \sim \cdots \sim \underbrace{\begin{bmatrix} 1 & -3 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 16 \end{bmatrix}}_{\text{3 pivots positions}}$$

Circle correct conclusion: Matrix A *is* / *is not* invertible.

EXAMPLE: Suppose H is a 5×5 matrix and suppose there is a vector **v** in $\mathbf{R}^5$ which is not a linear combination of the columns of H. What can you say about the number of solutions to $H\mathbf{x} = \mathbf{0}$?

Solution Since **v** in $\mathbf{R}^5$ is not a linear combination of the columns of H, the columns of H do not ___________ $\mathbf{R}^5$.

So by the Invertible Matrix Theorem, $H\mathbf{x} = \mathbf{0}$ has

____.

Invertible Linear Transformations

For an invertible matrix A,

$$A^{-1}A\mathbf{x} = \mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbf{R}^n$$

and

$$AA^{-1}\mathbf{x} = \mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbf{R}^n.$$

Pictures:

A linear transformation $T : \mathbf{R}^n \to \mathbf{R}^n$ is said to be **invertible** if there exists a function $S : \mathbf{R}^n \to \mathbf{R}^n$ such that

$$S(T(\mathbf{x})) = \mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbf{R}^n$$

and

$$T(S(\mathbf{x})) = \mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbf{R}^n.$$

Theorem 9

Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible if and only if A is an invertible matrix. In that case, the linear transformation S given by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the unique function satisfying

$$S(T(\mathbf{x})) = \mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbf{R}^n$$

and

$$T(S(\mathbf{x})) = \mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbf{R}^n.$$