1. Math 235 Review Questions

The following questions should be accessible to any student who is enrolled in M235 in Fall 2001, and almost all of these questions have appeared on M235/M136 final examinations over the last 8 years. These questions have been choosen without prejudice, that is to say we do not make any claim that they are representative of the questions which are on the final examination which you will write, however, if you can answer them all yourself then you will certainly be ready to attempt your final examination.

Conrad Hewitt will be in EL101 on Saturday Dec. 8th 2:00 p.m - 5:00 p.m. to help you go over any problems which are causing difficulty. In addition the Tutorial Centre will be open on Friday Dec. 7th. 9:00 a.m. - 6:00 p.m.

In your examination you will be asked to prove at least part of at least one of the following results from the course notes.

- Lemma 1.3
- Lemma 1.5
- Lemma 1.12
- Theorem 3.3
- Lemma 3.5
- Lemma 3.7
- Lemma 4.1
- Theorem 4.6
- Lemma 4.15

There is no True/False question on your final examination.
Your examination will concentrate mostly on the material which appeared after the midterm, however you will be expected both to know and to understand the earlier material.

Q1 Consider the linear transformation $T : R^4 \rightarrow R^3$, defined by

$$\begin{aligned} T((1,0,0,0)) &= (1,2,3) \\ T((0,1,0,0)) &= (0,-1,-2) \\ T((0,0,1,0)) &= (1,1,1) \\ T((0,0,0,1)) &= (2,5,8) \end{aligned}$$

a) Find a basis for the nullspace of T.
b) (i) State the dimension theorem .
b (ii) Apply the dimension theorem to the above T to find $rank(T)$.
c) Does the vector $\mathbf{v} = (1,0,-1)$ lie in the range of T?
Justify your answer.
Q2 a) (i) Let $T : V \rightarrow W$, be a linear transformation.
Define the nullspace, $N(T)$ and range, $R(T)$.
a) (ii) Prove that the range is a subspace of W.
b) Now let $T : V \rightarrow V$ be a linear operator. Prove that if T^2 is the zero transformation, then

$$R(T) \subseteq N(T).$$

Q3 Let $T : P_3(R) \rightarrow M_{2\times 2}(R)$ be the linear transformation defined by

$$T(f) = \begin{bmatrix} f(0) & f''(0) \\ f'(0) & f'''(0) - 2f'(1) \end{bmatrix}$$

a) Show that T is linear.
b) Let $\beta = \{1, x, x^2, x^3\}$and $\gamma = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$.
Find ${}_\gamma[T]^\beta$.
c) Use ${}_\gamma[T]^\beta$ to evaluate $T(\mathbf{v})$ where $\mathbf{v} = 1 + 2x + 3x^2 + 4x^3$.
Q4 a) Define an isomorphism between vector spaces V and W.
b) If $T : V \rightarrow W$, and $\beta = \{u_1, ..., u_r\}$ is a linearly independent subset of V, the prove that $T(\beta)$ is a linearly independent subset of W.
c) Let U be a finite-dimensional subspace of V. Use the result of part b) to prove that $dim(U) = dimT((U))$.
Q5 Let $T : V \rightarrow W$ be a linear transformation from the vector space V to the vector space W.
a) Define what it means for T to be one-to-one.

b) Define what it means for T to be onto.
c) Let V and W have the same dimension. Use the dimension theorem to show that T is one-to-one iff T is onto.
Q6 Let $T : P_2(R) \to R^3$ be the linear transformation defined by

$$T(p(x)) = (p(0), p'(0) - p(1), p''(1)).$$

a) Find a basis for the nullspace of T.
b) Find a basis for the range of T.
c) State the dimension theorem and verify that it holds for this T.
d) Determine whether or not T is one-to-one.
e) Determine whether or not T is onto.
Q7 Let $T : P_1(R) \to M_{2\times2}(R)$ and $S : M_{2\times2}(R) \to R^2$ be defined by:

$$T(f(x)) = \begin{bmatrix} f(0) + f(1) & f'(0) \\ 2f'(0) & f(0) - f(1) \end{bmatrix}, \qquad S\begin{bmatrix} a & b \\ c & d \end{bmatrix} = (a + d, b - c).$$

Let α, β and γ be the following ordered bases for $P_1(R), M_{2\times2}(R)$ and R^2 respectively:

$$\alpha = \{1, x\}, \beta = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right\}, \gamma = \{(1,0), (0,1)\}.$$

a) Find ${}_\beta[T]^\alpha$. b) Find ${}_\gamma[S]^\beta$.
c) Using the results of parts a) and b), find ${}_\gamma[ST]^\alpha$
Q8 Consider the system of equations:

$$\begin{aligned} x - y &= 1 \\ 2x + 5y + z &= -4 \\ -x + 3y + 4z &= -2 \end{aligned}$$

a) Write down the coefficient matrix of this system and calculate its determinant.
b) Why does the system have a unique solution (Hint use part a)).
c) Solve the system using Cramer's rule. You may use the facts that

$$\det\begin{bmatrix} 1 & -1 & 1 \\ 2 & 5 & -4 \\ -1 & 3 & -2 \end{bmatrix} = 5, \text{and } \det\begin{bmatrix} 1 & 1 & 0 \\ 2 & -4 & 1 \\ -1 & -2 & 4 \end{bmatrix} = -23.$$

Q9 a) Let A be a $n \times n$ real matrix. Show that $det(AA^t) \geq 0$.
b) Let B be an invertible $n \times n$ matrix. Show that $det(B^{-1}) = \frac{1}{\det(B)}$.
Q 10 a) Suppose that

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} = -1,$$

find

$$\det \begin{bmatrix} -2a & -2b & -2c \\ 2d+g & 2e+h & 2f+k \\ 3g & 3h & 3k \end{bmatrix}.$$

b) Suppose that A is an $n \times n$ matrix, and that the sum of the entries in each column of A is zero, prove that $det(A) = 0$.
c) If A, B and C are $n \times n$ matrices and ABC is invertible, show that B is invertible.
Q 11 a) Let A be a $n \times n$ real matrix, express $det(3A)$ in terms of $det(A)$.
b) Find the determinant of the $n \times n$ matrix B_n, where

$$B_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ c & 1 & 1 & \cdots & 1 & 1 \\ c & c & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c & c & c & \cdots & 1 & 1 \\ c & c & c & \cdots & c & 1 \end{bmatrix}.$$

Q12 Let A be a 2×2 matrix, and suppose $A^2 + A - 2I = 0_2$, then what are the possible choices for A?
Q13 a)For the following matrix A, find an orthogonal matrix P and a diagonal matrix D such that $P^{-1}AP = D$:

$$A := \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}.$$

b) Diagonalise the following matrix, B, over R, or explain why B cannot be diagonalised.

c) Diagonalise the following matrix, B, over C, or explain why B cannot be diagonalised.

$$B = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix}.$$

Q14 Let $\langle \ , \ \rangle$ be the standard inner product on C^n, Let A be a real symmetric matrix in $M_{n\times n}(C)$.
a) Show for the definitions that $\langle A\mathbf{v}, \mathbf{w}\rangle = \langle \mathbf{v}, A\mathbf{w}\rangle$, for all $\mathbf{v}$,$\mathbf{w}$ C^n.
b) Show that every eigenvalue, λ, of A is real.
c) Show that if $\mathbf{v}$ and $\mathbf{w}$ are eigenvectors of A corresponding to distinct eigenvalues λ , μ respectively, then $\mathbf{v}$ and $\mathbf{w}$ are orthogonal.
Q15 Consider the quadratic form

$$f(x, y, z) = x^2 - 8xy + 16y^2 - 3z^2.$$

a) Find a symmetric matrix A such that

$$(x, y, z)A(x, y, z)^T = f(x, y, z).$$

b) Orthogonally diagonalise the matrix A found in part a).
c) Express the quadratic form f in terms of new variables u, v, w, for which there are no cross terms, and relate the new variable to the old variables.
Q16 Let $T : V \to V$, be a linear operator on a finite-dimensional vector space V.
a) For $V = R^3$, and $T(a, b, c) = (2a + b - 2c, 2b, 2c)$, find the minimal polynomial $m_T(t)$ of T.
b) Prove in general that the minimal polynomial of T, $m_T(t)$ divides any polynomial, $p(t)$ that satisfies $p(T) = 0$, the zero operator.
c) (i) Let T be an operator on V such that $T^2 = T$, where T is neither the identy nor the zero operator. Find the minimal polynomial of T.
c) (ii) Prove that T is not invertible.
Q17 Let V be a real inner product space of dimension n, and let W be a proper non-zero subspace of V.
Let $T : V \to V$ be such that $T(\mathbf{v})$ is the orthogonal projection of $\mathbf{v}$ onto W.
a) Prove that $T^2 = T$.

b) Find the minimal polynomial of T.
c) Is T diagonalizable?
d) Is T invertible?
e) Assume that $dim(W) = k$, find the characteristic polynomial of T.
Q18 Prove that the real matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix},$$

is not diagonalisable.
Q19 a) State the Cayley-Hamilton Theorem.
b) Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$. Evaluate A^{100}.
Q20 The matrix $A = \begin{bmatrix} 7 & 4 & -5 \\ 4 & -2 & 4 \\ -5 & 4 & 7 \end{bmatrix}$ has eigenvalues 6, -6, 12.
Find an orthogonal matrix Q such that $Q^{-1}AQ$ is diagonal.
Q21Sketch the ellipse in the plane given by the equation

$$2x^2 - 4xy + 5y^2 = 36.$$

Q22 a)If A and B are orthogonal $n \times n$ matrices, show that AB is orthogonal.
b) Suppose A is an orthogonal matrix and it is lower triangular. Show that it is diagonal.
c) Suppose B is a real symmetric $n \times n$ matrix and $B^2 = B$. Show that $A = I - 2B$ is orthogonal and symmetric.
Q23 Diagonalise the following matrix $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \\ -1 & 0 & 5 \end{bmatrix}$.
Q24 Let T be a linear operator on a vector space V.
a) Define the following terms
(i) An eigenvector of T.
(ii) An eigenvalue of T.
b) Let $(\lambda, \mathbf{v})$ and$(\mu, \mathbf{w})$ be eigenpairs of T and suppose that $\lambda \neq \mu$. Prove that $\{\mathbf{v}, \mathbf{w}\}$ is linearly independent.
Q25 Let λ be an eigenvalue with associated eigenspace E_λ .
a) Define the algebraic multiplicity a_λ of λ .
b) Show that $1 \leq \dim(E_\lambda) \leq a_\lambda$.

c) Let $A = \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}$. Find a non-zero polynomial $p(t)$ such that $p(A) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Q26 a) Which of the following are diagonalisable over R?

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}.$$

b) Find the general solution of the differential equation:

$$\begin{aligned} \frac{dx}{dt} &= x + 2y \\ \frac{dy}{dt} &= 2x + y \end{aligned}$$

Q27 Let T be a linear operator on a vector space V.
a) (i) Define what it means for a subspace W of V to be T-invariant.
a) (ii) Show that the eigenspace E_λ of eigenvalue λ is T-invariant.
a (iii) Define the T-cyclic subspace generated by the vector $\mathbf{v} \in V$.
b) Let $\beta = \{v_1, v_2, v_3, v_4\}$ be a basis for V. Assume that:

$$\begin{aligned} T(v_1) &= 2v_1 \\ T(v_2) &= v_3, -2v_4 \\ T(v_3) &= 2v_1 - v_2 - 2v_4 \\ T(v_4) &= v_1 - v_2 - v_3 - v_4 \end{aligned}$$

(i) Find the T-cyclic subspace generated by v_2 .
(ii) Give another T-invariant subspace that contains v_2 .

Q28 Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix}$.

a) Solve the system of differential equations

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}.$$

b) Solve the initial value problem

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}(\mathbf{t}=0) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Q29 a) Let A be a $n \times n$ matrix. Define what it means for
(i) λ to be an eigenvalue of A.
(ii) $\mathbf{v}$ to be an eigenvector of A.
b) Answer the following questions providing a brief explanation.
(i) The 3×3 matrix A had eigenvalues $-2, 0, 5$. Is A diagonalisable?
(ii) The characteristic polynomial of matrix B is $-(t-1)^2(t+3)$.
The dimension of the eigenspace of eigenvalue 1 is 1. Is B diagonalisable?
(iii) The matrix C has eigenvalues 0 and 2. Is C invertible.
Q30 a) Let $h(x, y, z) = 2x^2 + xy + 2yx + z$. Find a symmetric matrix A such that $h(x, y, z) = (x, y, z)A(x, y, z)^T$.
b) Let $q(x, y, z) = (x, y, z)A(x, y, z)^T$, where $B = \begin{bmatrix} 8 & -2 & 2 \\ -2 & 5 & 4 \\ 2 & 4 & 5 \end{bmatrix}$.
Given that the characteristic polynomial of B is $-t^3 + 18t^2 - 81t$
(i) Show that the eigenvalues of B are 0 and 9.
(ii) Find an orthogonal matrix P and a symmetric matrix D such that $P^{-1}BP = D$.
Q31 Let A be a real $n \times n$ matrix, and assume that $A^2 = -I$. Prove that
a) A has no real eigenvalues.
b) n is even.
c) $det(A) = 1$.
Q32 Sketch the conic

$$2x^2 + 6xy + 10y^2 = 1,$$

in the x-y plane, clearly indicating its principal axes.

Q33 Let $A = \begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix}$.
a) Given that the characteristic polynomial of A is $(-3-t)(t^2+2t-3)$, determine whether or not A is diagonalisable.

b) Find the real numbers a, b, c such that

$$A^4 = aA^2 + bA + cI.$$

Q34 Let A be a real $n \times n$ matrix.
a) Give the definition of the characteristic polynomial, $p(t)$, of A.
b) Prove that A is singular iff 0 is an eigenvalue of A
c) Suppose that B is a real $n \times n$ non-singular matrix, and $\mathbf{x} \in R^n$.
Prove that $\mathbf{x}$ is an eigenvector of AB iff $B\mathbf{x}$ is an eigenvector of BA.
Q35 Let A be a real 3×3 matrix such that
(i) $trace(A) = 1$, (ii) $det(A) = -4$, (iii) 2 is an eigenvalue of A.
a) Find all the eigenvalues of A.
b) Prove that A is diagonalisable.

Q36 a) Orthogonally diagonalise the following matrix $A = \begin{bmatrix} 5 & 0 & 4 \\ 0 & -3 & 0 \\ 4 & 0 & -1 \end{bmatrix}$,
that is find an orthogonal matrix, P, such that $P^{-1}AP$ is diagonal.
b) Find another orthogonal matrix, Q, such that $Q^{-1}AQ$ is diagonal.