1 Notation

Why is "abbreviation" such a long word? *Anonymous*

1.1 General comments

As in linear programming, there are various alternative (but equivalent) formulations of the 'standard form' SDP. A very common format is the 'inequality form'

$$\begin{array}{ll} \text{minimize} & b^T y \\ \text{subject to} & \sum_{i=1}^m y_i A_i \preceq C, \end{array} \tag{1.1}$$

where the variable is $y \in I\!R^n$, and the problem data are the matrices $A_i, C \in \mathcal{S}^n$ and the vector b. The constraint

$$\sum_{i=1}^m y_i A_i \preceq C,$$

is called a *linear matrix inequality.*

Problem (1.1) is readily transformed into other equivalent forms used in the literature.

We can assume without loss of generality that the matrices A_i, $i = 1, \dots, m$, are linearly independent. Introducing a slack variable S we can express (1.1) as

$$\begin{array}{lrcrcl} \text{minimize} & b^T y & & & & \\ \text{subject to} & \sum_{i=1}^m A_i y_i & + & S & = & C \\ & & & S & \succeq & 0, \end{array} \tag{1.2}$$

with variables y and S. Since by assumption the matrices A_i are linearly independent, the set

$$\mathcal{V} = \left\{ S = C - \sum_{i=1}^m y_i A_i \;\middle|\; y \in I\!R^m \right\}$$

is an affine subset of $\mathcal{S}^n$ with dimension m. Therefore there exist symmetric matrices G_i, $i = 1, \dots, n(n+1)/2 - m$, and a vector d such that

$$\mathcal{V} = \{ S \in \mathcal{S}^n \mid G_i \bullet S = d_i,\ i = 1, \dots, n(n+1)/2 - m \}.$$

We can also compute a matrix G_0 that satisfies

$$G_0 \bullet A_i = -b_i, \quad i = 1, \dots, m,$$

by solving an underdetermined set of m equations in $n(n+1)/2$ variables, so that for $S = C - \sum_i A_i y_i$ we have

$$b^T y = G_0 \bullet S - G_0 \bullet C.$$

With those definitions we can express (1.2) as

$$\begin{array}{ll} \text{minimize} & G_0 \bullet S \\ \text{subject to} & G_i \bullet S = d_i, \quad i = 1, \dots, n(n+1)/2 - m \\ & S \succeq 0. \end{array}$$

As in linear programming, we can also consider SDPs with inequality and equality constraints and mixtures of restricted and free variables.

It is also common to have more than one matrix variable. More precisely we can consider problems of the form

$$\begin{array}{ll} \text{minimize} & \sum_{j=1}^{L} C_j \bullet X_j \\ \text{subject to} & \sum_{j=1}^{L} A_{i,j} \bullet X_j = b_i, \quad i = 1, \ldots, m \\ & X_j \succeq 0, \quad j = 1, \ldots, L, \end{array} \tag{1.3}$$

where the variables are $X_j \in \mathcal{S}^{n_j}$. if we define block-diagonal matrices

$$X = \operatorname{diag}(X_1, \ldots, X_L), \quad C = \operatorname{diag}(C_1, \ldots, C_L), \quad A_i = \operatorname{diag}(A_{i,1}, \ldots, A_{i,L}),$$

As an extreme case, we can have diagonal blocks of dimension one in (1.3), in which case the problem reduces to an LP.

In a similar way, we can consider SDPs in inequality form (1.1), with multiple linear matrix inequality constraints

$$\begin{array}{ll} \text{minimize} & b^T y \\ \text{subject to} & \sum_{i=1}^{m} y_i A_{i,j} \preceq C_j, \quad j = 1, \ldots, L. \end{array}$$

Again this is equivalent to (1.1) if we define block-diagonal matrices

$$C = \operatorname{diag}(C_1, \ldots, C_L), \quad A_i = \operatorname{diag}(A_{i,1}, \ldots, A_{i,L}).$$

Finally, we mention that some authors prefer the following compact formulation.

$$\begin{array}{lccc} \text{minimize} & C \bullet X & & \\ \text{subject to} & \mathcal{A}X & = & b \\ & X & \succeq & 0 \end{array}$$

where $\mathcal{A} : \mathcal{S}^n \to I\!R^m$ is a linear operator.

1.2 Overview

We conclude this introductory chapter by listing the most important notational conventions used in this book.

1. **Operations**

   - $\succeq$ the Löwner partial order on positive semidefinite matrices, $A \succeq B$ if $B - A$ is positive semidefinite.
   - $\succ$ the order on positive definite matrices, $A \succ B$ if $B - A$ is positive definite.
   - $\otimes$ the Kronecker product.
   - $\bullet$ the trace operator, i.e., $A \bullet B = \operatorname{Trace}(A^T B)$. Or

     $$\langle X, Y \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij} Y_{ij}.$$

     the standard inner product on two $(m \times n)$ real matrices.

2. **Sets**
   - $\mathbb{R}^n$ the n dimensional euclidean space.
   - $\mathbb{M}^{m\times n}$ or $\mathbb{R}^{m\times n}$ the space of all $m \times n$ matrices.
   - $\mathcal{S}^m$ the space of all $m \times m$ symmetric matrices.
   - $\mathcal{P} = \{X : X \succeq 0\}$ the convex cone of all symmetric and positive semidefinite matrices.
3. **Vectors**
   - x a vector.
   - x^T the transpose of the vector x.
   - x_i the ith component of the vector x.
   - $x^{(k)}$ the kth vector in a sequence of vectors.
   - $x^T y$ the inner-product of two vectors.
   - e_i the ith coordinate vector of R^n.
   - $1_n = e \in R^n$ the vector of all ones.
   - $I_n = \text{Diag}\,(1_n)$ is a unit matrix.
   - 0_n is the zero vector in R^n.
4. **Matrices**
   - A a matrix.
   - A^T the transpose of the matrix A.
   - A^k the kth matrix in a sequence of matrices.
   - $(A^k)^T$ the transpose of A^k.
   - $((A^k)^T)_{ij}$ the ijth entry of the transpose of A^k.
5. **Operators on matrices**
   (a) $\mathcal{A} : \mathcal{S}^n \longrightarrow \mathbb{R}^m$.
   (b) $\mathcal{A}^*$ the adjoint of the linear operator $\mathcal{A}$.
6. **Semidefinite programming problem**

   The matrices C, A_i are in $\mathcal{S}^n$.

$$\begin{array}{lccll} \text{minimize} & C \bullet X & & & \\ \text{subject to} & A_i \bullet X & = & b_i & \text{for all } i = 1, \cdots, m \\ & X & \succeq & 0 & \end{array}$$

   with its dual:

$$
\begin{array}{lrcl}
\text{maximize} & \sum_{i=1}^{m} b_i y_i & & \\
\text{subject to} & \sum_{i=1}^{m} A_i y_i + S & = & C \\
 & S & \succeq & 0
\end{array}
$$

Alternatively, SDP can be formulated as:

$$
\begin{array}{lrcl}
\text{minimize} & C \bullet X & & \\
\text{subject to} & \mathcal{A}X & = & b \\
 & X & \succeq & 0
\end{array}
$$

with its dual:

$$
\begin{array}{lrcl}
\text{maximize} & b^T y & & \\
\text{subject to} & \mathcal{A}^* y & \preceq & C
\end{array}
$$