A Study of Finding SDP bounds for Quadractic Assignment Problem

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1 Introduction

The quadratic assignment problems can be formulated as follows

$$(QAP) \quad \mu^* := \begin{array}{cc} \min & q(X) = \operatorname{trace}(AXB - 2C)X^t \\ \text{subject to} & X \in \Pi, \end{array}$$

where $\Pi := \{X : X \text{ is a permutation matrix}\}$. These problems are NP hard and thus can unlikely be solved directly. For this reason heuristic methods such as branch and bound are extensively studied, and one of the key spots is to find a good bound to measure and control the progress of these heuristic algorithms. However, up to now they have not been good enough to solve problems larger than order $n = 20$. However, recent success with other (0,1)-problems, such as the max-cut problem, using semidefinite relaxations lead us to try a similar approach for QAP.

Before going into further discussion, we first present some notation and preliminary results.

1.1 Preliminaries

We work in the space of $n \times n$ *symmetric matrices* denoted $\mathcal{S}^n$.

For symmetric matrices $M_1 \preceq M_2$ ($M_1 \prec M_2$) refers to the Löwner partial order, i.e. that $M_1 - M_2$ is negative semidefinite (negative definite, respectively); similar definitions hold for positive semidefinite and positive definite.

We use the *Kronecker product*, or tensor product, of two matrices, $A \otimes B$, when discussing the quadratic assignment problem QAP; $\operatorname{vec}(X)$ denotes the vector formed from the columns of the matrix X, while $\mathrm{Mat}\, x$ denotes the matrix formed from the vector x.

e is the vector of ones and e_i is the i-th unit vector; E denotes the matrix of ones. The set of matrices with row and column sums one is denoted

$$\mathcal{E} := \{X : Xe = X^t e = e\}.$$

The set of (0,1)-matrices is denoted

$$\mathcal{Z} := \{X : X_{ij}^2 = X_{ij},\ i, j = 1, \ldots, n\}.$$

The set of orthogonal matrices is denoted as $\mathcal{O}$, and the set of non-negtive matrices is denoted as $\mathcal{N}$.

Proposition 1.1

$$\Pi = \mathcal{O} \cap \mathcal{Z} = \mathcal{E} \cap \mathcal{Z}$$

From the above proposition, we can see that the feasible set of QAP can be expressed by different sets of quadratic constraints and thus these different sets can lead different semidefinite programming relaxations. In this project, we will give several semidefinite programming relaxations corresponding to different expressions of the feasible set of QAP. For each of the relaxations, we will discuss its perspective being efficiently solved by interior point methods. Some numerical testing results are included for comparing.

2 $\Pi = \mathcal{O} \cap \mathcal{Z}$

Using the orthogonal equivalence, we get the following equivalent program to QAP.

$$(QAP_E) \quad \begin{array}{ccc} \mu^* := & \min & q(X) = \operatorname{trace}(AXB - 2C)X^t \\ & \text{subject to} & XX^t = I \\ & & X_{ij}^2 - X_{ij} = 0, \quad \forall i, j. \end{array}$$

The semidefinite relaxation is

$$(DO) \quad \begin{array}{cc} \max & -\operatorname{trace} S - \alpha \\ \text{subject to} & L_Q + \operatorname{Arrow}(w) + \mathrm{B}^0\mathrm{Diag}(S) \succeq 0, \end{array}$$

where

$$L_Q := \begin{bmatrix} 0 & -\operatorname{vec}(C)^t \\ -\operatorname{vec}(C) & B \otimes A \end{bmatrix},$$

$$\text{Arrow}\,(w) := \begin{bmatrix} w_0 & -\frac{1}{2}w^t_{1:n^2} \\ -\frac{1}{2}w_{1:n^2} & \text{Diag}\,(w_{1:n^2}) \end{bmatrix}$$

and

$$\text{B}^0\text{Diag}\,(S) := \begin{bmatrix} 0 & 0 \\ 0 & I \otimes S \end{bmatrix}.$$

The dual of the relaxation is

$$(PO)\quad \begin{array}{cc} \min & \text{trace}\, L_Q Y \\ \text{subject to} & \text{arrow}\,(Y) = e_0 \\ & \text{b}^0\text{diag}\,(Y) = I \\ & Y \succeq 0, \end{array}$$

where *arrow* and b^0diag are the adjoint operator for *Arrow* and *BoDiag* respectively.

$$\text{arrow}\,(\cdot) := \text{diag}\,(\cdot) - (0, (\cdot_{0,1:n^2})^t.$$

For a $(n^2+1)\times(n^2+1)$ matrix Y, *BoDiag* is formed by ignoring the first row and column of Y and then summing the next n blocks of $n \times n$ submatrices.

Since Slater's hold for problem (DO), there is no duality gap for these pair of semidefinite programming problem and the problem (PO) is attained. However, the problem (PO) has no interior, so we have to use relative interior. The following known results give a relative interior for the feasible set of problem (PO), see e.g. [1] for details. We need to use the transportation matrix, denoted T, i.e. the $2n\times n^2$ (0,1)-matrix which defines the constraints that the row and column sums are 1.

$$T = \begin{bmatrix} e^t & 0 & \cdots & \cdots & 0 \\ 0 & e^t & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & e^t \\ I & I & \cdots & \cdots & I \end{bmatrix}. \tag{2.1}$$

Theorem 2.1 *Let $x = vec(X)$ and F_S be the feasible set of (SDP). Define the centroid point*

$$\hat{Y} := \frac{1}{n!}\sum_{X\in\Pi_n} \begin{bmatrix} 1 & x^t \\ x & xx^t \end{bmatrix}. \tag{2.2}$$

Then:

1. *$\hat{Y}$ has a 1 in the (1,1) position and n diagonal $n \times n$ blocks with diagonal elements $1/n$. The first row (and column) equal the diagonal. The rest of the matrix is made up of $n \times n$ blocks with all elements equal to $1/(n(n-1))$ except for the diagonal elements which are 0:*

$$\hat{Y} =$$

$$= \left[\begin{array}{c|c} 1 & \frac{1}{n}e^t \\ \hline \frac{1}{n}e & E \otimes \left(\frac{1}{n(n-1)}(E-I)\right) - I \otimes \left(\frac{1}{n(n-1)}E - \frac{1}{n-1}I\right) \end{array}\right]$$

2. $$\operatorname{rank}(\hat{Y}) = (n-1)^2 + 1$$

3. $$\operatorname{diag}\hat{Y} = \begin{pmatrix} 1 \\ \frac{1}{n}e \end{pmatrix}$$

4. $$\operatorname{trace}(\hat{Y}) = 1 + n$$

5. *The n^2+1 eigenvalues of $\hat{Y}$ are given in the vector*

$$(2, \frac{1}{n-1}e_{(n-1)^2}, 0e_{2n-1})$$

6. $$\mathcal{N}(\hat{Y}) = \left\{ \begin{pmatrix} -\frac{1}{n}e^t e \\ e \end{pmatrix} : e \in \mathcal{R}(T^t) \right\},$$

where $\mathcal{N}(\hat{Y})$ and $\mathcal{R}(T^t)$ are the null space of $\hat{Y}$ and range of T^t, respectively.

Although we can not solve the semidefinite relaxation in this section, but the theoritical results about its reletive interior turn out to be very useful in finding the interior point in the following section.

3 $\Pi = \mathcal{E} \cap \mathcal{Z}$

The equivalent problem to QAP is the following

$$\begin{array}{ll} \min & \operatorname{trace}(AXB^t - 2C)X^t \\ \text{s.t.} & Xe = X^te = e \\ & X_{ij}^2 = X_{ij} \text{ for } i,j = 1,\ldots,n \end{array} \tag{3.1}$$

Still this relaxation problem does not have interior point. A special structure of this problems is that it has linear constraints, and hence we can do some projection to reduce our problem on a smaller space. For detailed results, see e.g. [2].

Let the $n \times (n-1)$-matrix V be such that

$$V^te = 0, \quad V^tV = I_{n-1}.$$

$$v = \frac{e}{\|e\|}, \quad P = [v : V],$$

Then we have the following results.

Lemma 3.1 *Let X be $n \times n$ and Y be $(n-1) \times (n-1)$. Suppose that X and Y satisfy*

$$X = P \begin{bmatrix} 1 & 0 \\ 0 & Y \end{bmatrix} P^t. \tag{3.2}$$

Then X is a feasible solution to qap problem (3.1) if and only if Y satisfies

$$[vv^t + VYV^t]_{ij}^2 = [vv^t + VYV^t]_{ij} \quad i,j = 1,\ldots,n.$$

Moreover, X is orthogonal if and only if Y is orthogonal.

Proof. Expanding (3.2) we get the equivalent relation

$$X = vv^t + VYV^t. \tag{3.3}$$

Thus $Xe = vv^te = e$, and similarly $e^tX = e^t$. Next, note that by (3.3)

$$X_{ij}^2 = x_{ij} \quad i,j = 1,\ldots,n$$

is equivalent to

$$[vv^t + VYV^t]_{ij}^2 = [vv^t + VYV^t]_{ij} \quad i,j = 1,\ldots,n.$$

The proof of orthogonality is straitforward.

$\square$

A nice representation for V is as follows

$$V = \begin{pmatrix} y & y & \dots & y \\ 1+x & x & \dots & x \\ x & 1+x & \dots & x \\ \vdots & & \ddots & \vdots \\ x & x & \dots & 1+x \end{pmatrix},$$

where $x = -1/(n+\sqrt{n})$, $y = -1/\sqrt{n}$. We denote the rows and the columns of V by V_i, $i = 1, \dots, n$ and U_j, $j = 1, \dots, n-1$, respectively, i.e.

$$V = \begin{pmatrix} V_1 \\ \vdots \\ V_n \end{pmatrix} = (U_1 \dots U_{n-1}).$$

Let

$$\hat{A} := V^t A V, \ \hat{B} := V^t B V, \ \hat{C} := 2V^t C V,$$

and

$$\hat{D} := \frac{2}{n} V^t A e e^t B^t V + \hat{C}.$$

By the above lemma 3.1, we can get the following projected quadratic assignment problem, which is equivalent to qap problem (3.1)

$$\begin{array}{ll} \min & \operatorname{trace}[(\hat{A} Y \hat{B}^t + \hat{D}) Y^t] + \frac{(e^t A e)(e^t B e)}{n^2} + 2\frac{e^t C e}{n} \\ \text{subject to} & [vv^t + VYV^t]_{ij}^2 = [vv^t + VYV^t]_{ij} \quad i, j = 1, \dots, n \end{array} \tag{3.4}$$

Without considering the constant in the objective, we have the equivalent problem

$$\begin{array}{ll} \min & \operatorname{trace}[(\hat{A} Y \hat{B}^t + \hat{D}) Y^t] \\ \text{subject to} & [vv^t + VYV^t]_{ij}^2 = [vv^t + VYV^t]_{ij} \quad i, j = 1, \dots, n \end{array} \tag{3.5}$$

Theorem 3.1 *Let X and Y be related by (3.2). Then X solves qap problem (3.1) if and only if Y solves projected qap (3.5).*

We can find the lowerbound for problem (3.5) by solving the following semidefinite programming problem.

$$
\begin{array}{ll}
\max & t \\
\text{s.t.} & L_Q - t\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \sum_{i,j=1}^n \tau_{ij}\begin{bmatrix} \frac{1}{n^2}-\frac{1}{n} & (\frac{1}{n}-\frac{1}{2})V_i\otimes V_j \\ ((\frac{1}{n}-\frac{1}{2})V_i\otimes V_j)^t & (V_i\otimes V_j)^t(V_i\otimes V_j) \end{bmatrix} \succeq 0.
\end{array}
\tag{3.6}
$$

where

$$
L_Q = \begin{bmatrix} 0 & \frac{1}{2}\mathrm{vec}\,(\hat{D})^t \\ \frac{1}{2}\mathrm{vec}\,(\hat{D}) & \hat{B}\otimes\hat{A} \end{bmatrix}.
$$

The dual for (3.6) is

$$
\begin{array}{ll}
\min & \mathrm{trace}\,(L_Q Y) \\
\text{s.t.} & Y_{00} = 1 \\
& \mathrm{trace}\,(\begin{bmatrix} \frac{1}{n^2}-\frac{1}{n} & (\frac{1}{n}-\frac{1}{2})V_i\otimes V_j \\ ((\frac{1}{n}-\frac{1}{2})V_i\otimes V_j)^t & (V_i\otimes V_j)^t(V_i\otimes V_j) \end{bmatrix} Y) = 0, \quad i,j = 1,\ldots,n \\
& Y \succeq 0
\end{array}
\tag{3.7}
$$

where $Y \in S^{(n-1)^2+1}$. We now try to find a interior for the feasible set of the problem (3.7).

Theorem 3.2 *Let*

$$
\begin{pmatrix} 1 \\ \mathrm{vec}\,(X_k) \end{pmatrix}, \quad X_k \in \Pi \quad k = 1,\ldots,(n-1)^2+1
$$

be $(n-1)^2+1$ linearly independent vectors. Then for each such X_k, $k = 1,\ldots,(n-1)^2+1$,

$$
\begin{pmatrix} 1 \\ \mathrm{vec}\,(V^t X_k V) \end{pmatrix} (1, \mathrm{vec}\,(V^t X_k V)^t)
$$

is a feasible point of the problem (3.7). Moreover,

$$
\hat{Z} = \frac{1}{(n-1)^2+1}\sum_{k=1}^{(n-1)^2+1} \begin{pmatrix} 1 \\ \mathrm{vec}\,(V^t X_k V) \end{pmatrix} (1, \mathrm{vec}\,(V^t X_k V)^t)
$$

is an interior point of the feasible set of problem (3.7).

Proof. For any $k = 1, \ldots, (n-1)^2 + 1$,

$$
\begin{aligned}
V^t X_k V &= \begin{bmatrix} U_1^t X_k U_1 & \ldots & U_1^t X_k U_{n-1} \\ \vdots & \vdots & \vdots \\ U_{n-1}^t X_k U_1 & \ldots & U_{n-1}^t X_k U_{n-1} \end{bmatrix} \\
&= \begin{bmatrix} \operatorname{trace}(X_k) U_1 U_1^t & \ldots & \operatorname{trace}(X_k) U_{n-1} U_1^t \\ \vdots & \vdots & \vdots \\ \operatorname{trace}(X_k) U_1 U_{n-1}^t & \ldots & \operatorname{trace}(X_k) U_{n-1} U_{n-1}^t \end{bmatrix} \\
&= \begin{bmatrix} \operatorname{vec}(X_k)^t U_1 \otimes U_1 & \ldots & \operatorname{vec}(X_k)^t U_1 \otimes U_{n-1} \\ \vdots & \vdots & \vdots \\ \operatorname{vec}(X_k)^t U_{n-1} \otimes U_1 & \ldots & \operatorname{vec}(X_k)^t U_{n-1} \otimes U_{n-1} \end{bmatrix}.
\end{aligned}
$$

Therefore

$$\begin{pmatrix} 1 \\ \text{vec}\,(V^t X_k V) \end{pmatrix} = \begin{pmatrix} 1 \\ \text{vec}\,(X_k)^t U_1 \otimes U_1 \\ \vdots \\ \text{vec}\,(X_k)^t U_{n-1} \otimes U_1 \\ \vdots \\ \text{vec}\,(X_k)^t U_1 \otimes U_{n-1} \\ \vdots \\ \text{vec}\,(X_k)^t U_{n-1} \otimes U_{n-1} \end{pmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & (U_1 \otimes U_1)^t \\ \vdots & \vdots \\ 0 & (U_{n-1} \otimes U_1)^t \\ \vdots & \vdots \\ 0 & (U_1 \otimes U_{n-1})^t \\ \vdots & \vdots \\ 0 & (U_{n-1} \otimes U_{n-1})^t \end{bmatrix} \begin{pmatrix} 1 \\ \text{vec}\,(X_k) \end{pmatrix}.$$

Let

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & (U_1 \otimes U_1)^t \\ \vdots & \vdots \\ 0 & (U_{n-1} \otimes U_1)^t \\ \vdots & \vdots \\ 0 & (U_1 \otimes U_{n-1})^t \\ \vdots & \vdots \\ 0 & (U_{n-1} \otimes U_{n-1})^t \end{bmatrix}$$

and

$$H = \begin{bmatrix} 1 & \dots & 1 \\ \text{vec}\,(V^t X_1 V) & \dots & \text{vec}\,(V^t X_{(n-1)^2+1} V) \end{bmatrix}$$

. Then we have

$$H = Q \begin{bmatrix} 1 & \dots & 1 \\ \mathrm{vec}\,(X_1) & \dots & \mathrm{vec}\,(X_{(n-1)^2+1}) \end{bmatrix}.$$

Since $U_k,\ k = 1, \dots, n-1$ are orthogonal to each other with their norms being equal to one, $U_k \otimes U_l,\ k, l = 1, \dots, n-1$ are also orthogonal to each other, hence Q has full row rank. By the linear independency of $(1, \mathrm{vec}\,(X_k)^t),\ k = 1, \dots, (n-1)^2 + 1$, therefore, we have H is nonsingular. Therefore we have

$$\begin{aligned} \hat{Z} &= \tfrac{1}{(n-1)^2+1} \textstyle\sum_{k=1}^{(n-1)^2+1} \begin{pmatrix} 1 \\ \mathrm{vec}\,(V^t X_k V) \end{pmatrix} (1, \mathrm{vec}\,(V^t X_k V)^t) \\ &= \tfrac{1}{(n-1)^2+1} H H^t, \end{aligned}$$

which means $\hat{Z}$ is positive definite. Now we show that

$$\begin{pmatrix} 1 \\ \mathrm{vec}\,(V^t X_k V) \end{pmatrix} (1, \mathrm{vec}\,(V^t X_k V)^t)$$

is a feasible point for problem (3.7).

For any $i, j = 1, \dots, n$, let

$$W = \begin{bmatrix} (V_1 V_j^t)(V_1 V_i^t) & \dots & (V_n V_j^t)(V_1 V_i^t) \\ \vdots & \vdots & \vdots \\ (V_1 V_j^t)(V_n V_i^t) & \dots & (V_n V_j^t)(V_n V_i^t) \end{bmatrix}.$$

We can also rewrite Q as

$$Q = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & V_1 \otimes U_1^t & \dots & V_1 \otimes U_{n-1}^t \\ \vdots & \vdots & \vdots & \vdots \\ 0 & V_n \otimes U_1^t & \dots & V_n \otimes U_{n-1}^t \end{bmatrix}.$$

Then

$$\mathrm{trace}\,(\left[\begin{array}{cc} \frac{1}{n^2}-\frac{1}{n} & (\frac{1}{n}-\frac{1}{2})V_i\otimes V_j \\ ((\frac{1}{n}-\frac{1}{2})V_i\otimes V_j)^t & (V_i\otimes V_j)^t(V_i\otimes V_j) \end{array}\right]\left(\begin{array}{c} 1 \\ \mathrm{vec}\,(V^tX_kV) \end{array}\right)(1,\mathrm{vec}\,(V^tX_kV)^t))$$

$$= (1,\mathrm{vec}\,(X_k)^t)Q^t\left[\begin{array}{cc} \frac{1}{n^2}-\frac{1}{n} & (\frac{1}{n}-\frac{1}{2})V_i\otimes V_j \\ ((\frac{1}{n}-\frac{1}{2})V_i\otimes V_j)^t & (V_i\otimes V_j)^t(V_i\otimes V_j) \end{array}\right]Q\left(\begin{array}{c} 1 \\ \mathrm{vec}\,(X_k) \end{array}\right)$$

$$= (1,\mathrm{vec}\,(X_k)^t)\left[\begin{array}{cc} \frac{1}{n^2}-\frac{1}{n} & (\frac{1}{n}-\frac{1}{2})\mathrm{vec}\,(W)^t \\ ((\frac{1}{n}-\frac{1}{2})\mathrm{vec}\,(W) & \mathrm{vec}\,(W)\mathrm{vec}\,(W)^t \end{array}\right]\left(\begin{array}{c} 1 \\ \mathrm{vec}\,(X_k) \end{array}\right)$$

$$= \frac{1}{n^2}-\frac{1}{n}+(\frac{2}{n}-1)\mathrm{vec}\,(X_k)^t\mathrm{vec}\,(W)+(\mathrm{vec}\,(X_k)^t\mathrm{vec}\,(W))^2$$

$$= \frac{1}{n^2}-\frac{1}{n}+(\frac{2}{n}-1)\mathrm{trace}\,(X_kW)+(\mathrm{trace}\,(X_kW))^2$$

$$= 0,$$

by noticing that

$$V_kV_l^t = \begin{cases} -\frac{1}{n} & \text{if } k\neq l \\ 1-\frac{1}{n} & \text{if } k=l \end{cases}$$

It is straitforward to see that $\hat{Z}$ is an interior point for problem (3.7).

□Also

by noticing that the columns of V are orthogonal to each other, we can easily have the following results.

Theorem 3.3

$$\sum_{i,j=1}^{n}\left[\begin{array}{cc} \frac{1}{n^2}-\frac{1}{n} & (\frac{1}{n}-\frac{1}{2})V_i\otimes V_j \\ ((\frac{1}{n}-\frac{1}{2})V_i\otimes V_j)^t & (V_i\otimes V_j)^t(V_i\otimes V_j) \end{array}\right] = \left[\begin{array}{cc} 1-n & 0 \\ 0 & I \end{array}\right]$$

From the above two theorems, we can easily get the interior point for both the problem (3.6) and problem (3.7), respectively. Note that these interior points have nothing to do with A, B and C, hence, we have actually found interior points which are independent on the data of problem. Also, we can easily check that interior point $\hat{Z}$ satisfies

$$\mathrm{b}^0\mathrm{diag}\,(\hat{Z}) = I,$$

which corresponds to the orthogonality constraints.

	OPT	NB0	PRJ
nug05	50	49	49
nug06	86	74	74
nug08	214	179	179

Table 1: Preliminary Numerical Results

4 Numerical Results and Conclusion

We use the matlab code of Franz Rendl to test our projected semidefinite relaxation problems. We used Nugent's problems to do the numerical test. First, we tested the relaxation problems which do not include the constraints derived from the orthogonality of Y. The lowerbounds are very bad. Second, we tested the relaxation problems which include the constraints corresponding to the redundant constraints $Y'Y = I$. The lowerbounds improve but still are far below the current lowerbounds for the test problems. Then we add the constraints corresponding to the redundant constraints $YY' = I$. This time, the lowerbounds are comparable with the current lowbounds, see table 1. where OPT is the optimal sulution, NB0 is the results in [1] without adding nonnegativity constraints and PRJ is the lowerbound given by our relaxation. Because of the projection, our relaxation problem have a very dense constraint matrix, which prevent us from testing larger problems. Therefore, getting rid of some redundant constraints becomes essential in designing an efficient algorithm. On the other hand, From the numerical results, we can see that redundant orthogonal constraints play an important role in improving the lowerbound. Note that those redundant orthogonal constraints can not be derived from combining linearly the current constraints, therefore, some kind of "cutting-plane" type algorithms are deserved to be a future study.

References

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[2] F. Rendl S.W. Hadley and H. Wolkowicz. A new lower bound via projection for the quadratic assignment problem. *Mathematics of Operations Research*, 17, 1992.