On a positive semidefinite relaxation of the cut polytope

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Abstract

We study the convex set $\mathcal{L}_n$ defined by

$$\mathcal{L}_n := \{X \mid X = (x_{ij}) \text{ positive semidefinite } n \times n \text{ matrix}, x_{ii} = 1 \text{ for all } i\}.$$

We describe several geometric properties of $\mathcal{L}_n$. In particular, we show that $\mathcal{L}_n$ has 2^{n-1} vertices, which are its rank one matrices, corresponding to all bipartitions of the set $\{1, 2, \ldots, n\}$.

Our main motivation for investigating the convex set $\mathcal{L}_n$ comes from combinatorial optimization, namely from approximating the max-cut problem. An important property of $\mathcal{L}_n$ is that, due to the positive semidefinite constraints, one can optimize over it in polynomial time. On the other hand, $\mathcal{L}_n$ still inherits the difficult structure of the underlying combinatorial problem. In particular, it is NP-hard to decide whether the optimum of the problem

$$\min \mathrm{Tr}(CX), X \in \mathcal{L}_n$$

is reached at a vertex. This result follows from the complete characterization of the matrices C of the form $C = bb^t$ for some vector b, for which the optimum of the above program is reached at a vertex.

1 Introduction

Let $\mathcal{L}_n$ denote the set of $n \times n$ symmetric positive semidefinite matrices $X = (x_{ij})$ which satisfy $x_{ii} = 1$ for all $i = 1, \ldots, n$. Thus,

$$\mathcal{L}_n := \{X \mid X \succeq 0, x_{ii} = 1, i = 1, \ldots, n\}.$$

The matrices belonging to $\mathcal{L}_n$ are called *correlation matrices*, see [12, 17]. For a symmetric matrix $X = (x_{ij})$, let $\tau(X) := (x_{ij})_{1 \le i < j \le n}$ denote the $\binom{n}{2}$-vector which is the upper triangular part of X. We set

$$\widetilde{\mathcal{L}}_n := \{\tau(X) \mid X \in \mathcal{L}_n\}.$$

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Hence, the convex body $\widetilde{\mathcal{L}}_n$ is the projection of $\mathcal{L}_n$ on the $\binom{n}{2}$-dimensional subspace. As an example, see the body $\widetilde{\mathcal{L}}_3$ depicted in Fig. 1. We call the convex body $\widetilde{\mathcal{L}}_n$ an *elliptope* (coming from *ellip*soid and poly*tope*).

The set $\mathcal{L}_n$ of correlation matrices has been studied in several papers [3], [12], [17], [18]. There is considered mainly the question of determining what is the possible rank of a correlation matrix which is an extreme point of $\mathcal{L}_n$. (Recall that a point is an *extreme point* of $\mathcal{L}_n$ if it cannot be expressed as a proper convex combination of other points of $\mathcal{L}_n$.) It is shown that $\mathcal{L}_n$ has extreme points of rank k if and only if $k(k+1) \leq 2n$ (see [17] for a short proof). A formula for the dimension of the faces of $\mathcal{L}_n$ is also given in [17]; see the section on faces below.

Here, we are interested in the vertices of $\mathcal{L}_n$, i.e., in the extreme points of $\mathcal{L}_n$ having a full dimensional normal cone. We show that $\mathcal{L}_n$ has 2^{n-1} vertices which are, namely, its rank one matrices. In other words, the vertices of $\mathcal{L}_n$ are the matrices DJD, where D is a diagonal matrix with ±1-entries and J is the all ones matrix. So, up to permutation of the rows and columns, the vertices of $\mathcal{L}_n$ are of the form

$$\left(\begin{array}{c|c} 1 & -1 \\ \hline -1 & 1 \end{array}\right).$$

As $\mathcal{L}_n$ is a nonpolyhedral convex set, it has extreme points that are not vertices. For instance, the correlation matrix

$$\begin{pmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{pmatrix} \tag{1}$$

is an extreme point of $\mathcal{L}_3$, but is not a vertex as its normal cone has dimension 1.

This paper is motivated by a 'hard' combinatorial optimization problem, namely, the *maximum cut* problem (abbreviated as *max-cut*):

$$\max_{x \in \{0,1\}^n} \sum_{1 \leq i < j \leq n} c_{ij} |x_i - x_j|, \tag{2}$$

where $c \in \mathbb{R}^{\binom{n}{2}}$. The max-cut problem is well known to be equivalent with the *discrete 01-quadratic programming* problem:

$$\max_{x \in \{0,1\}^n} x^t Q x, \tag{3}$$

where Q is an $n \times n$ symmetric matrix. Since the exact optimum of the max-cut problem (or of the discrete quadratic programming problem) cannot be found efficiently unless $NP = P$, it is of interest to find good and efficient approximation procedures. Recently, it has been shown that optimization over the convex body $\widetilde{\mathcal{L}}_n$ provides a very good approximation for the max-cut problem. Moreover, the optimum is attained at a vertex of $\widetilde{\mathcal{L}}_n$ if and only if the max-cut problem is solved exactly.

The goal of our paper is to study the geometrical properties of the elliptope $\widetilde{\mathcal{L}}_n$, in order to understand better the structure of the optimization problem over it.

Actually, the elliptope $\widetilde{\mathcal{L}}_n$ displays an example of an interesting complexity phenomenon. Namely, the *weak optimization* problem over $\widetilde{\mathcal{L}}_n$ (see Section 3 for a definition) is polynomial since, by the theory of [13], checking whether a matrix belongs to $\mathcal{L}_n$ can be done efficiently, but testing whether the optimum is reached at a vertex of $\widetilde{\mathcal{L}}_n$ is NP-hard. (This phenomenon may be explained as follows. Suppose we want to optimize a linear objective function $c^t x$ over $\widetilde{\mathcal{L}}_n$. Assume that $x_0 \in \widetilde{\mathcal{L}}_n$ is an optimal solution. The polynomial time algorithm which solves the weak optimization problem provides us an approximation $\bar{x}_0 \in \widetilde{\mathcal{L}}_n$, so that $|c^t x_0 - c^t \bar{x}_0|$ is smaller than the prescribed error. However, this does not give us any information about the distance of the points x_0 and $\bar{x}_0$ which may be large. Moreover, $\bar{x}_0$ approximates not only the actual optimum, but also all the near-optimal solutions. Thus, the knowledge of $\bar{x}_0$ may not always be sufficient to recognize whether x_0 is a vertex.)

Contents of the paper.

In Section 2, we describe some basic geometric properties of $\mathcal{L}_n$ and $\widetilde{\mathcal{L}}_n$. Namely, we provide the formulas for polars, normal cones, and faces of $\mathcal{L}_n$ and $\widetilde{\mathcal{L}}_n$. As a consequence, we show that $\mathcal{L}_n$ has 2^{n-1} vertices; they are the rank one matrices of $\mathcal{L}_n$, corresponding to all bipartitions of the set $\{1, 2, \ldots, n\}$. In Section 3, we study the optimization problem

$$\min_{X \in \mathcal{L}_n} \mathrm{Tr}(CX). \tag{4}$$

Since $\mathcal{L}_n$ is not polyhedral, the optimum need not be attained in a vertex of $\mathcal{L}_n$. We call a symmetric matrix C *exact* if the optimum of (4) is attained in a vertex of $\mathcal{L}_n$. We give a complete characterization of the exact matrices of the form $C = bb^t$ for some vector b. In Section 4, we explain in more detail the connection with the approximation of the max-cut problem.

Some related topics.

By definition, the convex set $\mathcal{L}_n$ is nothing but a section of the cone PSD_n formed with the intersection by the hyperplanes $x_{ii} = 1$ for all i. The cone PSD_n has been extensively studied in the literature; see e.g.[2, 9, 16] for results on its faces. As a matter of fact, $\mathcal{L}_n$ inherits some of the good properties of PSD_n. However, its structure is much more complicated than that of PSD_n. For instance, the description of the faces of $\mathcal{L}_n$ follows from that of the faces of PSD_n (see Proposition 2.7). But, while each face of PSD_n is isomorphic to some PSD_r (for some $0 \leq r \leq n$), faces of $\mathcal{L}_n$ may be more complicated (see the example of an elliptic face of $\mathcal{L}_4$ in Example 2.11). The description of the faces of $\mathcal{L}_n$ leads to the interesting question of characterizing the subspaces that can be realized as kernels of matrices from $\mathcal{L}_n$.

The study of the set $\mathcal{L}_n$ of correlation matrices is also closely related to that of Euclidian distance matrices, which has an extensive literature (see, e.g., [15]). A symmetric matrix $X = (x_{ij})$ is called a *Euclidian distance matrix* if $x_{ij} = \| v_i - v_j \|^2$ for some vectors $v_1, \ldots, v_n \in \mathbb{R}^k$ $(k \geq 1)$. A well known result by Schoenberg [23] asserts that X is a Euclidian distance matrix if and only if the $(n-1) \times (n-1)$ matrix $P = (p_{ij})$ defined by

$$p_{ij} = \frac{1}{2}(x_{in} + x_{jn} - x_{ij})$$

for $1 \leq i, j \leq n-1$, is positive semidefinite. Equivalently, X is a Euclidian distance matrix if and only if $x = \tau(X)$ belongs to the cone NEG_n, called *negative type cone*, and defined by

$$\mathrm{NEG}_n = \{x \in \mathbb{R}^{\binom{n}{2}} \mid \sum_{1 \leq i < j \leq n} b_i b_j x_{ij} \leq 0 \text{ for all } b \in \mathbb{R}^n \text{ with } \sum_{1 \leq i \leq n} b_i = 0\}.$$

The supporting cone (see definition in (6)) of $\widetilde{\mathcal{L}}_n$ at each of its vertices is, up to symmetry, the cone NEG_n, i.e., the cone PSD_{n-1} (up to linear bijection) (see Remark 2.6).

Notation.

Given two $n \times n$ matrices $A = (a_{ij})$, $B = (b_{ij})$, we set

$$\langle A, B \rangle = \sum_{i,j=1}^{n} a_{ij} b_{ij}. \tag{5}$$

If A, B are symmetric, then we have the identity $\langle A, B \rangle = \mathrm{Tr}(AB)$. Recall that $\mathrm{Tr}(A) = \sum_{1 \leq i \leq n} a_{ii}$ denotes the *trace* of the matrix A. We write $A \succeq 0$ if A is a symmetric positive semidefinite matrix, i.e., $x^t A x \geq 0$ for all $x \in \mathbb{R}^n$. Let SYM_n (resp. PSD_n, DIAG_n) denote the set of all $n \times n$ symmetric (resp. symmetric positive semidefinite, diagonal) matrices. For $x \in \mathbb{R}^n$, $diag(x)$ denotes the diagonal matrix with diagonal entries $x_1, \ldots, x_n$ and, for a matrix A, $diag(A)$ denotes the vector consisting of the diagonal entries of A.

Let V be a real inner product space, whose inner product is denoted by $\langle , \rangle$. We will consider here two cases for the space V. When dealing with the set $\mathcal{L}_n$ of correlation matrices, V is the space SYM_n of symmetric matrices equipped with the inner product from (5). When dealing with its $\binom{n}{2}$-projection, namely, with the elliptope $\widetilde{\mathcal{L}}_n$, V is $\mathbb{R}^{\binom{n}{2}}$ equipped with the usual scalar product.

Let K be a convex set in V. The *polar* K^* of K is defined by

$$K^* = \{x \in V \mid \langle x, y \rangle \leq 1 \text{ for all } y \in K\}.$$

If K is a convex cone, then its polar coincides with the set $\{x \in V \mid \langle x, y \rangle \leq 0 \text{ for all } y \in K\}$. Given a boundary point x_0 of K, its *normal cone* $N(K, x_0)$ is defined by

$$N(K, x_0) = \{c \in V \mid \langle c, x \rangle \leq \langle c, x_0 \rangle \text{ for all } x \in K\}.$$

In other words, a vector $c \in V$ belongs to the normal cone at x_0 if the hyperplane $\langle c, x \rangle = \langle c, x_0 \rangle$ is supporting for K at x_0. So, the normal cone at x_0 consists of the normals to the supporting hyperplanes for K at x_0. The dimension of the normal cone allows one to classify the boundary points of K. Namely, a boundary point x_0 is a *vertex* of K if its normal cone is full dimensional, and x_0 is a *regular* (or *smooth*) point of K if $N(K, x_0)$ has dimension 1, i.e., there is only one supporting hyperplane for K passing through x_0. The *supporting cone* $C(K, x_0)$ at x_0 is then defined by

$$C(K, x_0) = \{x \in V \mid \langle c, x \rangle \leq 0 \text{ for all } c \in N(K, x_0)\}. \tag{6}$$

A subset F of K is called a *face* (or *extreme set*) of K if, for all $x \in F, y, z \in K$, $0 \leq \alpha \leq 1$, $x = \alpha y + (1-\alpha) z$ implies that $y, z \in F$. The set F is called an *exposed set* if $F = K \cap H$ for some supporting hyperplane H for K. Clearly, each exposed set is a face.

The convex sets considered in this paper are $\mathcal{L}_n$ and its projection $\widetilde{\mathcal{L}}_n$ and also some affine images $\mathcal{J}_n$ and $\widetilde{\mathcal{J}}_n$. Let J denote the all ones matrix. We recall the precise definitions:

$$\mathcal{J}_n = \{Y = (y_{ij}) \in \mathrm{SYM}_n \mid \frac{1}{2}J - Y \succeq 0,\ y_{ii} = 0 \text{ for all } i = 1, \ldots, n\},$$

$$\mathcal{L}_n = \{X = (x_{ij}) \in \mathrm{SYM}_n \mid X \succeq 0,\ x_{ii} = 1 \text{ for all } i = 1, \ldots, n\},$$

$$\widetilde{\mathcal{J}}_n = \tau(\mathcal{J}_n), \text{ and } \widetilde{\mathcal{L}}_n = \tau(\mathcal{L}_n).$$

Hence, $\mathcal{L}_n$ is the image of $\mathcal{J}_n$ under the linear bijection $X = J - 2Y$. Note that $\widetilde{\mathcal{L}}_n$ and $\widetilde{\mathcal{J}}_n$ are convex bodies (i.e., compact full dimensional convex sets) in $\mathbb{R}^{\binom{n}{2}}$. Clearly, $\widetilde{\mathcal{J}}_n$ and $\widetilde{\mathcal{L}}_n$ can be alternatively described by

$$\widetilde{\mathcal{J}}_n = \{y \in \mathbb{R}^{\binom{n}{2}} \mid \sum_{1 \leq i < j \leq n} b_i b_j y_{ij} \leq \frac{1}{4}(\sum_{1 \leq i \leq n} b_i)^2, \text{ for all } b \in \mathbb{R}^n\},$$

$$\widetilde{\mathcal{L}}_n = \{x \in \mathbb{R}^{\binom{n}{2}} \mid \sum_{1 \leq i < j \leq n} b_i b_j x_{ij} \geq -\frac{1}{2} \sum_{1 \leq i \leq n} b_i^2, \text{ for all } b \in \mathbb{R}^n\}.$$

Given a subset S of $\{1, \ldots, n\}$, let J_S and L_S denote the $n \times n$ symmetric matrices defined by

$$(J_S)_{ij} = \begin{cases} 1 & \text{if } i \in S, j \notin S \text{ or } i \notin S, j \in S \\ 0 & \text{otherwise,} \end{cases}$$

$$(L_S)_{ij} = \begin{cases} -1 & \text{if } i \in S, j \notin S \text{ or } i \notin S, j \in S \\ 1 & \text{otherwise.} \end{cases}$$

for $1 \leq i, j \leq n$. So, up to permutation of the rows and columns, they are of the form: $J_S = \left(\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right)$, $L_S = \left(\begin{array}{c|c} 1 & -1 \\ \hline -1 & 1 \end{array}\right)$. Hence, $J_S \in \mathcal{J}_n$, $L_S = J - 2J_S \in \mathcal{L}_n$, $L_S = L_{\{1,\ldots,n\}\setminus S}$, $J_\emptyset = 0$, and $L_\emptyset = J$. We call the matrices J_S,L_S a 01-cut matrix and a ± 1-cut matrix, respectively; we refer to J_S,L_S as the *cut matrices*.

Finally, let us already introduce some of the main notions used in the paper. Given a vector $b \in \mathbb{R}^n$, its *gap* $\gamma(b)$ is defined by

$$\gamma(b) := \min_{S \subseteq \{1,\ldots,n\}} |\sum_{i \in S} b_i - \sum_{i \in \{1,\ldots,n\}\setminus S} b_i|.$$

The vector b is said to be *balanced* if

$$|b_i| \leq \sum_{1 \leq j \leq n,\ j \neq i} |b_j| \text{ for all } i = 1, \ldots, n.$$

A subspace V of $\mathbb{R}^n$ is called *realizable* if V is contained in the null space of some matrix of $\mathcal{L}_n$. The orthogonal complement of V is denoted by $V^\perp$. For $x, y \in \mathbb{R}^n$, $x \circ y$ denotes the vector $z \in \mathbb{R}^n$ whose i-th entry is equal to $x_i y_i$. We let e denote the all ones vector.

2 Geometry of the elliptope

The main result of this section is the characterization of the vertices of the elliptope $\widetilde{\mathcal{L}}_n$. As tools for this result, we describe the polar of $\widetilde{\mathcal{L}}_n$ and the normal cone at any point of $\widetilde{\mathcal{L}}_n$. We also present results on the faces of $\widetilde{\mathcal{L}}_n$ and full treatment in the case of $\widetilde{\mathcal{L}}_3$.

Let us first observe that the bodies $\mathcal{L}_n$ and $\widetilde{\mathcal{L}}_n$ have some symmetries. Given a subset A of $\{1, \ldots, n\}$, let D_A denote the $n \times n$ diagonal matrix whose i-th diagonal entry is equal to 1 if $i \in A$ and to -1 otherwise. The mapping $Sw_A : \mathrm{SYM}_n \longrightarrow \mathrm{SYM}_n$, defined by

$$Sw_A(X) = D_A X D_A,$$

is called *switching*. Observe that $Sw_A(L_S) = L_{S \triangle A}$ for each subset S and $Sw_A(\mathcal{L}_n) = \mathcal{L}_n$. (Here, $S \triangle A = (A \setminus S) \cup (S \setminus A)$ denotes the symmetric difference of the sets A and S.) There is an obvious analogue of switching for the body $\widetilde{\mathcal{L}}_n$.

Polar.

Let W_n denote the set of all symmetric $n \times n$ matrices X with diagonal entries equal to 1. Then, we have the equality

$$\mathcal{L}_n = \mathrm{PSD}_n \cap W_n. \tag{7}$$

Hence, the inclusion $\mathrm{PSD}_n^* + W_n^* \subseteq \mathcal{L}_n^*$ holds trivially. In fact, equality holds as shown in the next result. Note that

$$\mathrm{PSD}_n^* = -\mathrm{PSD}_n,\ W_n^* = \{D \in \mathrm{DIAG}_n \mid \mathrm{Tr}(D) \leq 1\}.$$

(Recall that the polar is defined within the space of symmetric matrices.) Hence, the set $\{D - M \mid M \succeq 0,\ D \in \mathrm{DIAG}_n, \mathrm{Tr}(D) \leq 1\}$ is contained in $\mathcal{L}_n^*$ and it can be easily checked that it coincides with the set $\{D - M \mid M \succeq 0,\ D \in \mathrm{DIAG}_n, \mathrm{Tr}(D) = 1\}$.

PROPOSITION 2.1 *(i)* $\mathcal{L}_n^* = \{D - M \mid M \succeq 0,\ D \in \mathrm{DIAG}_n, \mathrm{Tr}(D) = 1\}$.
(ii) $\widetilde{\mathcal{L}}_n^* = \mathrm{Conv}(-2\tau(bb^t) \mid b \in \mathbb{R}^n, \| b \| = 1)$.

PROOF. (i) Let $Y \in \mathcal{L}_n^*$. We show that $Y \in \{D - M \mid M \in \mathrm{PSD}_n,\ D \in \mathrm{DIAG}_n, \mathrm{Tr}(D) = 1\}$. If not, then, for each $D \in \mathrm{DIAG}_n$ with $\mathrm{Tr}(D) = 1$, the matrix $D - Y$ is not positive semidefinite, i.e., $\lambda_{\min}(D - Y) < 0$. For $d \in H := \{x \in \mathbb{R}^n \mid \sum_{1 \leq i \leq n} x_i = 1\}$, set

$$f(d) := \lambda_{\min}(diag(d) - Y).$$

Set $\mu := \min_{1 \leq i \leq n} Y_{ii}$. Note that $f(d) = \min_{\|x\|=1} x^t(diag(d) - Y)x \leq \min_{1 \leq i \leq n}(d_i - \mu)$ (the last inequality follows by choosing for x the unit vectors). Hence, if $\| d \|_\infty \longrightarrow \infty$, then $\min_i d_i \longrightarrow -\infty$, which implies that $f(d) \longrightarrow -\infty$. This shows that $\sup_{d \in H} f(d) = \sup_{d \in H,\ \|d\|_\infty \leq M} f(d) = \max_{d \in H,\ \|d\|_\infty \leq M} f(d)$ for some constant M. Therefore, the optimum is attained in $\sup_{d \in H} f(d)$, i.e.,

$$\max(\lambda_{\min}(D - Y) \mid D \in \mathrm{DIAG}_n, \mathrm{Tr}(D) = 1) < 0.$$

The following result is shown in [4]: Let D_0 be the diagonal matrix with trace one for which the above maximization problem attains its optimum and set $\lambda_0 := \lambda_{\min}(D_0 - Y) < 0$. Then, there exists a set of vectors $v_1, \ldots, v_k$ which are eigenvectors of $D_0 - Y$ for the eigenvalue λ_0 and such that all diagonal entries of the matrix $X := \sum_{h=1}^{k} v_h v_h^t$ are equal to 1. Hence, the matrix X belongs to $\mathcal{L}_n$ and

$$\mathrm{Tr}((Y - D_0)X) = \sum_{h=1}^{k} \mathrm{Tr}((Y - D_0)v_h v_h^t) = -\sum_{h=1}^{k} \lambda_0 \mathrm{Tr}(v_h v_h^t) = -\lambda_0 \mathrm{Tr}(X) = -n\lambda_0.$$

Therefore, $\langle Y, X \rangle = \langle D_0, X \rangle + \langle Y - D_0, X \rangle = 1 - n\lambda_0 > 1$, contradicting the fact that $Y \in \mathcal{L}_n^*$.
(ii) Given $y \in \mathbb{R}^{\binom{n}{2}}$, let Y denote the $n \times n$ symmetric matrix whose upper triangular part is y and with diagonal entries equal to $-\frac{1}{n}$. Note that $y \in \widetilde{\mathcal{L}}_n^*$ if and

only if $Y \in \mathcal{L}_n^*$. Indeed, $\langle X, Y\rangle = 2x^t y - 1$ for all $X \in \mathcal{L}_n$ and $x = \tau(X)$. By (i), $Y \in \mathcal{L}_n^*$ if and only if Y is of the form: $Y = D - \sum_{h=1}^{k} \lambda_h b_h b_h^t$, where D is a diagonal matrix with trace 1, $b_1, \ldots, b_k$ are unit vectors, and $\lambda_1, \ldots, \lambda_k$ are nonnegative scalars; moreover, $\sum_{1\leq h\leq k} \lambda_h = 2$, since $-1 = \mathrm{Tr}(Y) = \mathrm{Tr}(D) - \sum_h \lambda_h$. Therefore, $y \in \widetilde{\mathcal{L}}_n^*$ if and only if y is of the form: $y = -\sum_{1\leq h\leq k} \lambda_h \tau(b_h b_h^t)$, where $b_1, \ldots, b_k$ are unit vectors and $\lambda_1, \ldots, \lambda_k$ are nonnegative scalars with $\sum_{1\leq h\leq k} \lambda_h = 2$. This shows that $y \in \widetilde{\mathcal{L}}_n^*$ if and only if $y \in \mathrm{Conv}(-2\tau(bb^t) \mid b \in \mathbb{R}^n, \| b \| = 1)$. ∎

Remark 2.2 As suggested by a referee, Proposition 2.1 (i) can also be proved using duality of semidefinite programming. Namely, given $Y \in \mathcal{L}_n^*$, consider the programs:

$$\begin{array}{ll} \max & \langle Y, X\rangle \\ \text{s.t.} & diag(X) = e \\ & X \succeq 0 \end{array} \tag{8}$$

$$\begin{array}{ll} \min & x^t e \\ \text{s.t.} & diag(x) - Y \succeq 0. \end{array} \tag{9}$$

The programs (8) and (9) are dual programs and, hence, have the same optimum value, equal to $\alpha \leq 1$, as $Y \in \mathcal{L}_n^*$. If x is an optimum solution to the program (9), then $diag(x) - Y$ is a positive semidefinite matrix, which yields the desired decomposition for Y. (Note that, in order to show that strong duality holds for the programs (8) and (9), one has to check that a technical condition, the so-called Slater condition, holds. Details can be found, e.g., in [20].) ∎

Normal cone.

Proposition 2.3 *Let $A \in \mathcal{L}_n$ and $a = \tau(A) \in \widetilde{\mathcal{L}}_n$. Then*
(i) $N(\mathcal{L}_n, A) = \{D - M \mid D \in DIAG_n, M \succeq 0, \langle M, A\rangle = 0\}$.
(ii) $N(\widetilde{\mathcal{L}}_n, a) = \mathrm{Cone}(-\tau(bb^t) \mid b \in \mathrm{Ker}(A))$.

Proof. (i) First, if $D \in \mathrm{DIAG}_n$ and $M \succeq 0$ with $\langle M, A\rangle = 0$, then $D - M \in N(\mathcal{L}_n, A)$ since, for all $X \in \mathcal{L}_n$, $\langle D - M, X\rangle = \mathrm{Tr}(D) - \langle M, X\rangle \leq \mathrm{Tr}(D) = \langle D - M, A\rangle$. Conversely, let $Y \in N(\mathcal{L}_n, A)$, i.e., Y is a symmetric matrix satisfying $\langle Y, X\rangle \leq \langle Y, A\rangle$ for all $X \in \mathcal{L}_n$. We show that Y can be decomposed as $Y = D - M$, where $D \in \mathrm{DIAG}_n$, $M \succeq 0$, and $\langle M, A\rangle = 0$. We can suppose that the diagonal entries of Y are all equal to 0; indeed, $Y - diag(Y)$ also belongs to $N(\mathcal{L}_n, A)$, as $\langle diag(Y), X\rangle = \mathrm{Tr}(Y)$ holds for all $X \in \mathcal{L}_n$. Suppose first that $\langle Y, A\rangle = 0$. We show that $-Y$ is positive semidefinite, i.e., that $\langle Y, X\rangle \leq 0$ for all $X \in \mathrm{PSD}_n$. If

$X \in \mathcal{L}_n$, then $\langle Y, X\rangle \le 0$ holds by the assumption that $Y \in N(\mathcal{L}_n, A)$. If $X \succeq 0$ with $x_{ii} \le 1$ for all $1 \le i \le n$, then $X' := X + \mathrm{diag}(1 - x_{11}, \ldots, 1 - x_{nn}) \in \mathcal{L}_n$. Hence, $\langle Y, X'\rangle \le 0$, implying that $\langle Y, X\rangle \le 0$. Finally, if $X \succeq 0$, let α be a positive scalar such that the diagonal entries of αX are less than or equal to 1. By the previous case, $\langle Y, \alpha X\rangle \le 0$ which implies that $\langle Y, X\rangle \le 0$. We now suppose that $a := \langle Y, A\rangle \ne 0$. Then, $a > 0$ since $0 = \langle Y, I\rangle \le \langle Y, A\rangle$. Hence, $a^{-1}Y \in \mathcal{L}_n^*$. Therefore, by Proposition 2.1, $Y = D - M$ for some diagonal matrix D with trace a and $M \succeq 0$. Moreover, $\langle M, A\rangle = \langle D, A\rangle - \langle Y, A\rangle = \mathrm{Tr}(D) - a = 0$. This concludes the proof of (i).
(ii) Applying (i), we obtain that $N(\widetilde{\mathcal{L}}_n, a) = \{-\tau(M) \mid M \succeq 0,\ \langle M, A\rangle = 0\}$. The result follows since, for a decomposition $M = \sum_{1\le h\le k} b_h b_h^t$ of M as a sum of rank one matrices, $\langle M, A\rangle = 0$ holds if and only if $Ab_h = 0$, i.e., $b_h \in \mathrm{Ker}(A)$, for all h.
∎

Remark 2.4 Let us remark that, for $n = 3$, the normal cone at each point $\tau(L_S)$ of $\widetilde{\mathcal{L}}_n$ is a circular cone. By symmetry, it suffices to check this fact for the cut matrix $L_\emptyset = J$. For this, let us consider the section of the normal cone $N(\widetilde{\mathcal{L}}_n, \tau(J))$ by the hyperplane with equation $x_{12} + x_{13} + x_{23} = 3$. Note that the point $c = (1, 1, 1)$ belongs to $N(\widetilde{\mathcal{L}}_n, \tau(J)) \cap H$. One can easily check that each extreme ray $-\tau(bb^t)$ of $N(\widetilde{\mathcal{L}}_n, \tau(J))$ intersects H in a point which is at constant distance $\sqrt{6}$ from c. This shows, therefore, that $N(\widetilde{\mathcal{L}}_n, \tau(J))$ is a circular cone. We show in Fig. 2 the normal cone at a vertex. ∎

Vertices.

We can now characterize the vertices of $\widetilde{\mathcal{L}}_n$.

Theorem 2.5 *$\widetilde{\mathcal{L}}_n$ has 2^{n-1} vertices, namely, the vectors $\tau(L_S)$, for $S \subseteq \{1, \ldots, n\}$.*

Proof. We first check that each vector $\tau(L_S)$ is a vertex of $\widetilde{\mathcal{L}}_n$. Indeed, for $1 \le i < j \le n$, the hyperplane $x_{ij} = 1$ (resp. $-x_{ij} = 1$) is supporting for $\widetilde{\mathcal{L}}_n$ at $\tau(L_S)$ if $i, j \in S^2 \cup (\{1, \ldots, n\} \setminus S)^2$ (resp. if $i \in S, j \notin S$ or vice versa). This shows that the normal cone of $\widetilde{\mathcal{L}}_n$ at $\tau(L_S)$ is full dimensional, i.e., that $\tau(L_S)$ is a vertex of $\widetilde{\mathcal{L}}_n$. Conversely, let $A \in \mathcal{L}_n$ and suppose that $a = \tau(A)$ is a vertex of $\widetilde{\mathcal{L}}_n$. Then, there exist $\binom{n}{2}$ vectors $b_1, \ldots, b_{\binom{n}{2}}$ in the kernel of A such that the system $(\tau(b_i b_i^t) \mid 1 \le i \le \binom{n}{2})$ is linearly independent. Consider the $\binom{n}{2} \times \binom{n}{2}$ matrix M whose rows are the vectors $\tau(b_i b_i^t)$ and the submatrix M_1 formed by its first $n - 1$ columns, indexed by the pairs $(1, j)$ for $2 \le j \le n$. Then, M_1 has rank $n - 1$ and, thus, contains an $(n-1) \times (n-1)$ nonsingular submatrix which is indexed, say, by the vectors $b_1, \ldots, b_{n-1}$. It is easy to check that the vectors $b_1, \ldots, b_{n-1}$

are linearly independent. This shows that the matrix A has rank one and, thus, $A = aa^t$ for some $a \in \mathbb{R}^n$. But, $a \in \{-1,1\}^n$ since the diagonal entries $(a_i)^2$ of A are all equal to 1. Therefore, A is one of the cut matrices L_S. ∎

In particular, the vectors $\tau(L_S)$ are the only ± 1-valued members of $\widetilde{\mathcal{L}}_n$ (indeed, every ± 1-member of $\widetilde{\mathcal{L}}_n$ has a full dimensional normal cone, i.e., is a vertex of $\widetilde{\mathcal{L}}_n$).

REMARK 2.6 As a consequence of Proposition 2.3, we have the following assertions.
(i) The regular points of $\widetilde{\mathcal{L}}_n$, i.e., having a normal cone of dimension one, are of the form $\tau(A)$ for $A \in \mathcal{L}_n$ whose kernel $\mathrm{Ker}(A)$ has dimension 1.
(ii) Given $A \in \mathcal{L}_n$, the supporting cone of $\widetilde{\mathcal{L}}_n$ at the point $a = \tau(A)$ is given by $C(\widetilde{\mathcal{L}}_n, a) = \{x \in \mathbb{R}^{\binom{n}{2}} \mid \sum_{1 \leq i < j \leq n} b_i b_j x_{ij} \geq 0 \text{ for all } b \in \mathrm{Ker}(A)\}$.
In particular, the supporting cone at the vertex $\tau(L_\emptyset)$ coincides with the cone $-\mathrm{NEG}_n$ (i.e., is a linear bijective image of PSD_{n-1}) and the suporting cone at any other vertex $\tau(L_S)$ is a symmetric image of it, namely, $C(\widetilde{\mathcal{L}}_n, \tau(L_S)) = Sw_S(-\mathrm{NEG}_n)$. ∎

Faces.

We now turn to the description of the faces of $\mathcal{L}_n$. Let us first recall some well known facts about the faces of the cone PSD_n of positive semidefinite matrices. Let $A, B \in \mathrm{PSD}_n$ and let $\Phi(A)$ denote the smallest face of PSD_n containing A. Then, $B \in \Phi(A)$ if and only if $\mathrm{Ker}(A) \subseteq \mathrm{Ker}(B)$. Moreover, if A has rank r, then $\Phi(A)$ is isomorphic to PSD_r and, hence, has dimension $\binom{r+1}{2}$ [16]. In particular, every face of PSD_n is of the form

$$\Phi_V := \{X \in \mathrm{PSD}_n \mid V \subseteq \mathrm{Ker}(X)\}$$

for some subspace V of $\mathbb{R}^n$ and, conversely, Φ_V is a face of PSD_n for each subspace V.

Consider now the convex set $\mathcal{L}_n$. For $A \in \mathcal{L}_n$, let $F(A)$ denote the smallest face of $\mathcal{L}_n$ containing A. It is shown in [17] that, if A has rank r and if A is the Gram matrix of the vectors $v_1, \ldots, v_n \in \mathbb{R}^r$, then the dimension of the face $F(A)$ is equal to

$$\binom{r+1}{2} - \dim(v_i v_i^t \mid 1 \leq i \leq n).$$

(Compare with the fact that the dimension of $\Phi(A)$ in PSD_n is equal to $\binom{r+1}{2}$.) As an example, consider the matrix A from (1); it has rank 2 and it is an extreme point of $\mathcal{L}_3$. Note, indeed, that A is the Gram matrix of the vectors $v_1 := (1,0)$,

$v_2 := (-\frac{1}{2}, \frac{\sqrt{3}}{2})$, $v_3 := (-\frac{1}{2}, -\frac{\sqrt{3}}{2})$ and that the matrices $v_1 v_1^t$, $v_2 v_2^t$ and $v_3 v_3^t$ are linearly independent. The faces of $\mathcal{L}_n$ can, in fact, be defined in a simple way from the faces of PSD_n. We first recall a well known result.

If K_1, K_2 are two convex bodies and F_1, F_2 are faces of K_1, K_2, respectively, then $F_1 \cap F_2$ is a face of $K_1 \cap K_2$ and, moreover, each face of $K_1 \cap K_2$ arises in this way.

As noted in relation (7), $\mathcal{L}_n$ is the intersection of PSD_n and W_n. Clearly, W_n is the only face of W_n. Hence, we have the following result.

PROPOSITION 2.7 *Let $A, B \in \mathcal{L}_n$ and let $F(A)$ denote the smallest face of $\mathcal{L}_n$ containing A. Then, $B \in F(A)$ if and only if $Ker\,(A) \subseteq Ker(B)$. In particular, every face of $\mathcal{L}_n$ is of the form*

$$F_V := \{X \in \mathcal{L}_n \mid V \subseteq Ker(X)\}$$

for some subspace V of $\mathbb{R}^n$ and $F(A) = F_{Ker(A)}$ for $A \in \mathcal{L}_n$. ∎

COROLLARY 2.8 *Every face of $\mathcal{L}_n$ is exposed.*

PROOF. Consider a face F_V where V is a subspace of $\mathbb{R}^n$. Let $b_1, \ldots, b_k$ be an orthogonal base of V. Then, for $X \in \mathcal{L}_n$, $X \in F_V$ if and only if $b_i^t X b_i = 0$ for all $i = 1, \ldots, k$ or, equivalently, $\sum_{1 \le i \le k} b_i^t X b_i = 0$. Hence, the face F_V arises as the intersection of $\mathcal{L}_n$ by the supporting hyperplane $\sum_{1 \le i \le k} b_i^t X b_i = 0$. This shows that F_V is exposed. ∎

Note that, given a subspace V of $\mathbb{R}^n$, there always exists $X \succeq 0$ such that $V \subseteq \mathrm{Ker}(X)$, but there may exist no such $X \in \mathcal{L}_n$. This is the case, for instance, if $V \subseteq \mathbb{R}^2$ is generated by the vector $(2, 1)$. For this reason, we call a subspace V of $\mathbb{R}^n$ *realizable* if there exists $X \in \mathcal{L}_n$ such that $V \subseteq \mathrm{Ker}(X)$. Clearly, the only realizable subspaces of dimension $n-1$ are the kernels of the cut matrices L_S. We give in the next section the characterization of the 1-dimensional realizable subspaces. The characterization of the realizable subspaces of dimension k, $2 \le k \le n-2$, is an open problem (already posed in [12]).

We give some further results on the faces of $\mathcal{L}_n$. In particular, we show that the convex segment joining any two vertices of $\mathcal{L}_n$ is a face of $\mathcal{L}_n$. We describe all faces of $\mathcal{L}_3$ and we indicate how every face of $\mathcal{L}_n$ can be "lifted" to a face of $\mathcal{L}_{n+1}$.

PROPOSITION 2.9 *Let A, B be distinct subsets of $\{1, \ldots, n\}$. Then, the convex segment $[L_A, L_B] = \{\alpha L_A + (1-\alpha) L_B \mid 0 \le \alpha \le 1\}$ is a face of $\mathcal{L}_n$.*

PROOF. Using the switching symmetry, we can suppose that $B = \emptyset$. We show that the segment $[L_\emptyset, L_A]$ is a face of $\mathcal{L}_n$. Set $Y := \frac{1}{2}(L_\emptyset + L_A)$. Then, $\mathrm{Ker}(Y) = \{b \in \mathbb{R}^n \mid \sum_{i\in A} b_i = \sum_{i\notin A} b_i = 0\}$. One can easily check that a symmetric $n \times n$ matrix matrix X belongs to $F(Y)$ if and only if there exists a scalar a such that $|a| \leq 1$ and $X = (x_{ij})$ with $x_{ij} = 1$ for $i, j \in A$ or $i, j \notin A$ and $x_{ij} = a$ for $i \in A, j \notin A$. In other words, $X \in F(Y)$ if and only if X is the convex combination $\frac{a+1}{2} L_\emptyset + \frac{1-a}{2} L_A$ of $L_\emptyset$ and L_A. This shows that $[L_\emptyset, L_A] = F(Y)$ is a face of $\mathcal{L}_n$. ∎

Note that there exist faces of $\mathcal{L}_n$ of dimension 2 that are not polyhedral. We describe such a face for $\mathcal{L}_4$ in Example 2.11 below. We now present a full description of the faces of the body $\mathcal{L}_3$.

PROPOSITION 2.10 *Every proper face of $\mathcal{L}_3$ is, either reduced to a single point of $\mathcal{L}_3$, or is an edge (1-dimensional face) joining two vertices of $\mathcal{L}_3$ (there are six such faces).*

PROOF. Let F_V be a face of $\mathcal{L}_3$, where V is a (realizable) subspace of $\mathbb{R}^3$. If $\dim(V) = 2$, then F_V is reduced to a vertex of $\mathcal{L}_3$. Suppose now that V has dimension 1. Let $b \in V$. Then, by Proposition 3.2, b is balanced. We can suppose that $|b_1|, |b_2|, |b_3| \leq 1$ and, for instance, $b_1 = 1$. Then, $b = (1, \alpha, \beta)$ with $1 \leq |\alpha| + |\beta|$. Let $X \in \mathcal{L}_3$ be of the form $\begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix}$, where $x, y, z \in \mathbb{R}$. Then, $X \in F_V$ if and only if $Xb = 0$, i.e., x, y, z satisfy the system

$$\begin{cases} \alpha x + \beta y &= -1 \\ x + \beta z &= -\alpha \\ y + \alpha z &= -\beta. \end{cases} \tag{10}$$

The determinant of the system (10) is equal to $-2\alpha\beta$. If $\alpha\beta \neq 0$, then the system (10) has a unique solution (x, y, z), i.e., F_V consists of a single point of $\mathcal{L}_3$. If, say, $\alpha = 0$, then $\beta = \pm 1$. The solutions of the sytem (10) are of the form $(-\beta z, -\beta, z)$ for $z \in \mathbb{R}$. For $\beta = 1$, we obtain that $X \in F_V$ if and only if $X = \frac{z+1}{2} L_{\{1\}} + \frac{1-z}{2} L_{\{3\}}$ with $|z| \leq 1$ and, thus, $F_V = [L_{\{1\}}, L_{\{3\}}]$. Similarly, for $\beta = -1$, $X \in F_V$ if and only if $X = \frac{z+1}{2} L_\emptyset + \frac{1-z}{2} L_{\{2\}}$, i.e., $F_V = [L_\emptyset, L_{\{2\}}]$. ∎

EXAMPLE 2.11 Let V denote the 1-dimensional subspace of $\mathbb{R}^4$ spanned by the vector $b = (1, 1, 1, 0)$. One can check easily that a symmetric 4×4 matrix X belongs to F_V if and only if X is of the form $X = \begin{pmatrix} 1 & -1/2 & -1/2 & x \\ -1/2 & 1 & -1/2 & y \\ -1/2 & -1/2 & 1 & z \\ x & y & z & 1 \end{pmatrix}$, where $x, y, z \in \mathbb{R}$ satisfy $x + y + z = 0$ and $x^2 + xy + y^2 \leq \frac{3}{4}$ (the first condition ensures that $Xb = 0$ and the second one that $X \succeq 0$). Hence, F_4 is a 2-dimensional face of $\mathcal{L}_4$ with the shape of an ellipse. ∎

Finally, we present an operation which permits lifting each face of $\mathcal{L}_n$ to a face of $\mathcal{L}_{n+1}$. Let X be a symmetric $n \times n$ matrix with diagonal entries equal to 1, of the form

$$X = \left(\begin{array}{c|c} Y & a \\ \hline a^t & 1 \end{array} \right),$$

where $a \in \mathbb{R}^{n-1}$. Consider the $(n+1) \times (n+1)$ symmetric matrix X' defined by

$$X' = \left(\begin{array}{c|c|c} Y & a & a \\ \hline a^t & 1 & 1 \\ \hline a^t & 1 & 1 \end{array} \right).$$

Lemma 2.12 *$X \in \mathcal{L}_n$ if and only if $X' \in \mathcal{L}_{n+1}$.*

Proof. Let $y \in \mathbb{R}^{n-1}$, $x_n, x_{n+1} \in \mathbb{R}$ and set $x := (y, x_n)$, $x' := (y, x_n + x_{n+1}) \in \mathbb{R}^n$, $z := (y, x_n, x_{n+1})$ and $z' := (y, x_n, 0) \in \mathbb{R}^{n+1}$. Then, we have that $z^t X' z = x'^t X x'$ and $x^t X x = z'^t X' z'$. This shows that $X \succeq 0$ if and only if $X' \succeq 0$ and, thus, $X \in \mathcal{L}_n$ if and only if $X' \in \mathcal{L}_{n+1}$. ∎

Corollary 2.13 *Let F be a face of $\mathcal{L}_n$. Then, $F' := \{X' \mid X \in F\}$ is a face of $\mathcal{L}_{n+1}$.*

Proof. Let $X \in \mathcal{L}_n$ and let $F := F(X)$ be the smallest face of $\mathcal{L}_n$ containing X. We show that F' coincides with $F(X')$, the smallest face of $\mathcal{L}_{n+1}$ containing X'. The kernel of X' is spanned by the vectors $(b, 0)$ for $b \in \mathrm{Ker}(X)$ and $(0, \ldots, 0, 1, -1)$. Hence, if $Z \in F(X')$, then the n-th and $(n+1)$-th columns of Z coincide because $(0, \ldots, 0, 1, -1) \in \mathrm{Ker}(Z)$, and the submatrix of Z formed by its first n columns and rows belongs to $F(X)$ because $(b, 0) \in \mathrm{Ker}(Z)$ for all $b \in \mathrm{Ker}(X)$. Therefore, $Z = A'$ for some $A \in F(X)$. This shows that $F' = F(X')$ is a face of $\mathcal{L}_{n+1}$. ∎

3 Optimizing over the elliptope

Let us recall that a symmetric matrix C is called exact if the optimum of (4) is attained in a vertex of $\mathcal{L}_n$. By definition of the normal cone, C is exact if and only if

$$-C \in \bigcup_{S \subseteq \{1,\ldots,n\}} N(\mathcal{L}_n, L_S).$$

The motivation to study exact matrices comes from the application to the max-cut problem, which will be discussed in the next section. The main result of this section is the characterization of the exact matrices C which are of the form $C = bb^t$ for a vector b.

Let $b = (b_1, \ldots, b_n)$ be a vector. The *gap* of b, denoted as $\gamma(b)$, is defined as

$$\gamma(b) := \min_{S \subset \{1,\ldots,n\}} |b(S) - b(\overline{S})| \tag{11}$$

where $b(S) := \sum_{i \in S} b_i$. In particular, we have $\gamma(b) = 0$ if $b(S) = b(\overline{S})$ for some S. We say that a vector $b = (b_1, \ldots, b_n)$ is *balanced* if

$$|b_i| \leq \sum_{j=1}^{i-1} |b_j| + \sum_{j=i+1}^{n} |b_j| \tag{12}$$

for every $i = 1, \ldots, n$. In other words, a vector b is balanced if none of its entries (in absolute value) is larger than the sum of the remaining entries (in absolute value).

Given a pair of vectors x and y, let $x \circ y$ denote the vector $z = (z_i)$ with entries $z_i := x_i y_i$. Let $V^\perp$ denote the orthogonal complement of a linear subspace V. Let $e = (1, \ldots, 1)$ denote the vector of all ones.

Lemma 3.1 *A linear subspace V is realizable if and only if $e \in cone\{x \circ x \mid x \in V^\perp\}$.*

Proof. Assume that V is realizable, i.e., $V \subset \mathrm{Ker}(X)$ for some $X \in \mathcal{L}_n$. Since X is positive semidefinite, $\mathrm{Ker}(X)$ is the eigenspace of the minimum eigenvalue $\lambda_{\min} = 0$, and hence X can be written as $X = \sum_{i=1}^k x_i x_i^t$ where $x_1, \ldots, x_k \in (\mathrm{Ker}(X))^\perp \subset V^\perp$. Since $diag(X) = e$, we have $e = \sum_{i=1}^k x_i \circ x_i \subset cone\{x \circ x \mid x \in V^\perp\}$. Conversely, let $e = \sum_{i=1}^k x_i \circ x_i$ for some $x_1, \ldots, x_k \in V^\perp$. Set $X := \sum_{i=1}^k x_i x_i^t$. Obviously, $V \subset \mathrm{Ker}(X)$, and $X \in \mathcal{L}_n$ since X is positive semidefinite and $diag(X) = e$. ∎

The following result can be shown using Lemma 3.1; it was already proved in [5, Theorem 3.2].

Proposition 3.2 ([5]) *Let $b \in \mathbb{R}^n$ be a vector. Then the linear space $V = \langle b \rangle$ generated by b is realizable if and only if b is balanced.* ∎

THEOREM 3.3 *Let C be a matrix of the form $C = bb^t$ for some $b \in \mathbb{R}^n$. Then, C is exact if and only if one of the following holds:*
(i) b is balanced and has gap $\gamma(b) = 0$, or
(ii) b is unbalanced.

PROOF. We have $\langle bb^t, X\rangle = b^t X b \geq 0$ for every b and every $X \in \mathcal{L}_n$. Hence $\min_{X\in\mathcal{L}_n} b^t X b \geq 0$, with equality if and only if $b \in \mathrm{Ker}(X)$ for some $X \in \mathcal{L}_n$, i.e., if the subspace $\langle b\rangle$ is realizable. Hence, from Proposition 3.2, the minimum is equal to zero if and only if b is balanced. Assume that b is balanced. We claim that

$$bb^t \text{ is exact if and only if } \gamma(b) = 0. \tag{13}$$

The matrix bb^t is, by definition, exact if and only if $\min_{X\in\mathcal{L}_n} b^t X b$ is reached in a vertex L_S, i.e., $\langle bb^t, L_S\rangle = b^t L_S b = 0$ for some S. Hence $L_S b = 0$ since $L_S \succeq 0$. The latter is equivalent to $b(S) - b(\overline{S}) = 0$. Hence (13) is proved. Assume that b is not balanced. We claim that

$$bb^t \text{ is exact.} \tag{14}$$

Assume that all $b_i, i = 1,\ldots,n$, are nonnegative. Without loss of generality we may also assume that $b_1 > \sum_{i=2}^n b_i$. Define the vector $a = (a_i)$ by $a_1 = \sum_{i=2}^n b_i$ and $a_i = b_i$ for $i = 2,\ldots,n$. Hence, $a^t L_{\{1\}} a = 0$, which shows that the minimum of $\langle aa^t, X\rangle$ is reached in the vertex $L_{\{1\}}$ of $\mathcal{L}_n$. Hence, $-aa^t$ belongs to the normal cone of $L_{\{1\}}$,

$$-aa^t \in N(\mathcal{L}_n, L_{\{1\}}). \tag{15}$$

For every $i = 2,\ldots,n$, set $f_i = (1,0,\ldots,0,1,0,\ldots,0)$ (the 1- and i-th entries are equal to 1), and observe that $f_i \in \mathrm{Ker}(L_{\{1\}})$. Hence

$$-f_i f_i^t \in N(\mathcal{L}_n, L_{\{1\}}). \tag{16}$$

Now, it is easy to check that bb^t can be expressed as

$$bb^t = aa^t + \sum_{i=2}^n \lambda_i f_i f_i^t - diag(u),$$

where $\lambda_i = u_i := (b_1 - a_1)a_i \geq 0$ for $2 \leq i \leq n$ and $u_1 := -b_1(b_1 - a_1)$. Hence, (15) and (16) imply that $-bb^t \in N(\mathcal{L}_n, L_{\{1\}})$. Thus, bb^t is exact. In case that some b_i's are negative, apply switching with the set $S = \{i \mid b_i < 0\}$. ∎

For every $S \subseteq \{1,\ldots,n\}$, let $\mathcal{O}_S \subset \mathbb{R}^{\binom{n}{2}}$ denote the orthant

$$\mathcal{O}_S := \{x = (x_{ij}) \mid x_{ij} \geq 0 \text{ for } i,j \in S, \text{ or } i,j \notin S \text{ and } x_{ij} \leq 0 \text{ otherwise}\}.$$

Since $\widetilde{\mathcal{L}}_n$ is contained in the unit cube $[-1,1]^{\binom{n}{2}}$, we have

LEMMA 3.4 *For every S, the orthant $\mathcal{O}_S$ is entirely contained in the normal cone $N(\widetilde{\mathcal{L}}_n, \tau(L_S))$ of the vertex $\tau(L_S)$.* ∎

Let p_n denote the probability that a random vector $c \in \mathbb{R}^{\binom{n}{2}}$, with $\|c\| := \sum c_{ij}^2 = 1$, is exact.

COROLLARY 3.5 *We have $p_n \geq 2^{-\frac{1}{2}(n^2-3n+2)}$.*

PROOF. Since $\mathcal{O}_S \subset N(\widetilde{\mathcal{L}}_n, \tau(L_S))$ for every S by the above lemma, we have

$$p_n = Prob(c \in \bigcup_S N(\widetilde{\mathcal{L}}_n, \tau(L_S)) \geq Prob(c \in \bigcup_S \mathcal{O}_S).$$

Hence

$$p_n \geq \frac{2^{n-1}}{2^{\binom{n}{2}}} = 2^{-\frac{1}{2}(n^2-3n+2)}$$

since $\widetilde{\mathcal{L}}_n$ has 2^{n-1} vertices and the total number of orthants is $2^{\binom{n}{2}}$. ∎

In particular, we have $p_3 \geq 0.5$ by the corollary. Ch. Delorme (personal communication) computed the exact value $p_3 = 2(1 - \frac{1}{\sqrt{3}}) \simeq 0.845$.

Let us conclude this section with pointing out an interesting complexity aspect of the optimization over $\widetilde{\mathcal{L}}_n$:

(i) The weak optimization problem (WOPT) over $\widetilde{\mathcal{L}}_n$ is polynomial time solvable;

(ii) Testing whether the optimum over $\widetilde{\mathcal{L}}_n$ is attained at a vertex is NP-hard.

Let us recall that the *weak optimization problem* for a convex body K is defined in [13] as follows. Given a rational vector c and a rational number $\varepsilon > 0$, either (i) find a rational vector y such that $y \in S(K, \varepsilon)$ and $c^t x \leq c^t y + \varepsilon$ for all $x \in S(K, -\varepsilon)$, or (ii) assert that $S(K, -\varepsilon)$ is empty. (Here $S(K, \varepsilon)$ denote the set of points which lie in the ε-neighborhood of K; and $S(K, -\varepsilon)$ denote the set of points whose ε-neighborhood is contained in K.) The polynomial-time solvability of WOPT follows from the theory developed in [13], since one can efficiently check the (weak) membership of $X \in \mathcal{L}_n$ (by computing the minimum eigenvalue of X with sufficient precision, and inspecting its diagonal entries). On the other hand, the problem (ii) is NP-hard for $\widetilde{\mathcal{L}}_n$, as a corollary of Theorem 3.3. Given an integer vector $b = (b_1, \ldots, b_n)$, it is NP-hard to decide whether the gap $\gamma(b)$ is zero (cf. the *exact sum* problem in [10]). Thus, if we could decide whether or not the optimum of $x^t \tau(bb^t)$ is reached in a vertex of $\widetilde{\mathcal{L}}_n$, we would be able to solve the exact sum problem.

A practically efficient algorithm which can be used for the optimization problem over the elliptope $\widetilde{\mathcal{L}}_n$ is described in [22].

4 The max-cut problem

Let $G = (V, E)$ be a graph and let $c : E \to \mathbb{R}$ be an edge-weight function. The *max-cut problem* consists of finding a subset S of vertices for which the sum of the weights on the edges between S and $\bar{S}$ is maximum. Let us denote

$$mc(G, c) := \max_{S \subset V} \sum_{i \in S, j \notin S} c_{ij}.$$

The max-cut problem is polynomial-time solvable when G is planar ([14, 19]), and it is NP-complete for G general ([10]). Barahona and Mahjoub [1] introduced a polytope associated with the max-cut problem called the cut polytope. For our purpose, it is sufficient to recall the definition only for the case when $G = K_n$ is the complete graph. For a set $S \subset V$, let $\delta(S)$ (*the cut*) denote the edge set $\delta(S) := \{ij \mid i \in S, j \notin S\}$, and let $\chi^{\delta(S)}$ denote the characteristic vector of the cut $\delta(S)$ defined by $\chi_{ij}^{\delta(S)} = 1$ for $ij \in \delta(S)$, and $\chi_{ij}^{\delta(S)} = 0$ otherwise. The *cut polytope* P_n is defined as $P_n :=$ Conv$\{\chi^{\delta(S)} \mid S \subset V\}$. Hence, the max-cut problem can be alternatively defined as

$$mc(G, c) := \max_{x \in P_n} c^t x \tag{17}$$

The cut polytope P_n has been extensively studied; see, e.g., [7, 8].

A. Schrijver (personal communication) introduced the convex body $\tilde{\mathcal{J}}_n$ as a relaxation of the cut polytope. Indeed, since $\chi^{\delta(S)} = \tau(J_S) \in \tilde{\mathcal{J}}_n$ for every S, we have

LEMMA 4.1
(i) $P_n \subset \tilde{\mathcal{J}}_n$,
(ii) $mc(G, c) \leq \max_{x \in \tilde{\mathcal{J}}_n} c^t x$ *(extending c to all pairs of nodes by setting $c_{ij} = 0$ if the pair ij is not an edge of G).*

∎

Clearly, the symmetric matrix $C = (c_{ij})$ is exact if and only if the program $\max_{x \in \tilde{\mathcal{J}}_n} c^t x$ solves the max-cut problem.

For nonnegative weights c, the quality of the approximation of the max-cut by optimizing over the elliptope can be measured by the ratio

$$\frac{\max_{x \in \tilde{\mathcal{J}}_n} c^t x}{mc(G, c)}.$$

It has been conjectured in [5] that this ratio is bounded by 1.131 (with C_5 as the worst case). A recent result of [11] shows that this ratio is bounded by $\frac{1}{0.878} = 1.139$. Using a result of [5], one can prove a better bound for some special classes of weights.

PROPOSITION 4.2 *Let the weights c_{ij} be given by $c_{ij} = a_i a_j$ where $a_1, \ldots, a_n \in \mathbb{R}_+$. Then,*

$$\frac{\max_{x \in \widetilde{\mathcal{J}}_n} c^t x}{mc(G,c)} \leq \frac{9}{8} = 1.125.$$

PROOF. For $a_1, \ldots, a_n$ balanced, this result was shown in (Corollary 5.1,[5]). If $a_1, \ldots, a_n$ is not balanced, then C is exact by Theorem 3.3, implying that the ratio is equal to 1. ∎

Delorme and Poljak ([4, 5, 6]) considered earlier an eigenvalue upperbound $\varphi(G,c)$ for the max-cut $mc(G,c)$, defined as

$$\varphi(G,c) := \min_{u \in \mathbb{R}^n, \sum u_i = 0} \frac{n}{4} \lambda_{\max}(L(G,c) + diag(u))$$

where $L(G,c)$ denotes the Laplacian matrix of the weighted graph (G,c) and $\lambda_{\max}$ its maximum eigenvalue. (Recall that $L(G,c)$ is the $n \times n$ symmetric matrix defined by $L(G,c)_{ii} = \sum_{1 \leq j \leq n, j \neq i} c_{ij}$ and $L(G,c)_{ij} = -c_{ij}$ for $i \neq j$.) The computational experiments of [21] show that the eigenvalue bound provides a good approximation of the max-cut, since the relative error typically ranges between 1% – 5%.

Actually, the eigenvalue bound coincides with the semidefinite bound, i.e.,

$$\max_{x \in \widetilde{\mathcal{J}}_n} c^t x = \varphi(G,c),$$

as was shown by Poljak and Rendl [20], using duality of semidefinite programming.

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