Gap inequalities for the cut polytope

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Abstract

We introduce a new class of inequalities valid for the cut polytope, which we call gap inequalities. Each gap inequality is given by a finite sequence of integers, whose "gap" is defined as the smallest discrepancy arising when decomposing the sequence into two parts as equal as possible. Gap inequalities include the hypermetric inequalities and the negative type inequalities, which have been extensively studied in the literature. They are also related to a positive semidefinite relaxation of the max-cut problem.

A natural question is to decide for which integer sequences the corresponding gap inequalities define facets of the cut polytope. For this property, we present a set of necessary and sufficient conditions in terms of the root patterns and of the rank of an associated matrix. We also prove that there is no facet defining inequality with gap greater than one and which is induced by a sequence of integers using only two distinct values.

1 Introduction

Set $V := \{1, \dots, n\}$. Let $b = (b_1, \dots, b_n) \in \mathbb{Z}^n$ be a sequence of n integers. We define the **gap** $\gamma(b)$ of b by

(1.1) $$\gamma(b) := \min_S |b(S) - b(V \setminus S)|,$$

where $b(S) := \sum_{i \in S} b_i$ for any subset S of V. Equivalently,

$$\gamma(b) = \min_{x \in \{\pm 1\}^n} |x^T b|.$$

This notion of gap for a sequence $b \in \mathbb{Z}^n$ coincides with the notion of discrepancy considered in [LSV86] for arbitrary matrices; we specialize here the notion to the case of matrices having only one row.

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Computing the gap of an integer sequence is a hard problem. For instance, it is an NP-complete problem to decide if the gap is equal to zero. Indeed, the sequence b has gap zero if and only if it can be partitioned into two parts of equal weights. This is the partition problem, which is NP-complete; see [GJ79].

Given a sequence $b \in \mathbb{Z}^n$, we consider the following inequality in the $\binom{n}{2}$ variables x_{ij} $(1 \leq i < j \leq n)$:

$$(1.2) \qquad \sum_{1 \leq i < j \leq n} b_i b_j x_{ij} \leq \frac{\sigma(b)^2 - \gamma(b)^2}{4},$$

where $\sigma(b) := \sum_{1 \leq i \leq n} b_i$. The inequality (1.2) is called a **gap inequality**.

Our main motivation for introducing the inequalities (1.2) lies in their connection with the cut polytope CUT_n; indeed, they define valid inequalities for CUT_n. The following classes of gap inequalities have been extensively studied in the literature:

(i) The inequalities (1.2) with $\sigma(b) = 0$ (which implies that $\gamma(b) = 0$), known as the **negative type inequalities**.

(ii) The inequalities (1.2) with $\sigma(b) = 1$ (which implies that $\gamma(b) = 1$), known as the **hypermetric inequalities**.

Negative type inequalities were used by Schoenberg [Sch35, Sch38] for the characterization of the distance spaces that are isometrically ℓ_2-embeddable. Hypermetric inequalities were introduced by Deza [Dez61] and later, independently, by Kelly [Kel75] in connection with the study of ℓ_1-embeddable distance spaces. Among the hypermetric inequalities, large subclasses are known that define facets of the cut polytope; see, e.g., [DL92]. On the other hand, for the case $\sigma = 0$ of the negative type inequalities, the following is known.

PROPOSITION 1.3 [Dez62] *Every inequality* (1.2) *with* $\sigma(b) = 0$ *is implied by the inequalities* (1.2) *with* $\sigma(b) = 1$. □

In fact, using symmetries, this yields that every gap inequality with gap $\gamma = 0$ is implied by the gap inequalities with gap $\gamma = 1$. Therefore, no gap inequality with gap 0 defines a facet of the cut polytope.

Hence arises naturally the question of deciding what happens in the case $\gamma \geq 2$? So far, we have not been able to find any example of a gap inequality with $\gamma \geq 2$ and that defines a facet of CUT_n. This leads us to conjecture that none exists.

CONJECTURE 1.4 *For any integer sequence* $b \in \mathbb{Z}^n$*, if the inequality* (1.2) *defines a facet of the cut polytope* CUT_n*, then* $\gamma(b) = 1$.

In view of the remarks above, in order to prove Conjecture 1.4, it suffices to show that every gap inequality that defines a facet of CUT_n has gap $\gamma \in \{0,1\}$. In this paper, we give several results in connection with this conjecture.

The paper is organized as follows. In Section 2 we present some preliminary results. In particular, we explain how the gap inequalities (1.2) arise in connection with the cut polytope CUT_n and how they relate with the inequalities defining a positive semidefinite relaxation of CUT_n. We group in Section 3 several results on the gap. We present in Section 4 a characterization of the gap inequalities that define facets of the cut polytope, which is in terms of conditions on the possible root patterns (i.e., in the n-space rather than in the $\binom{n}{2}$-space, where the inequalities live). We show in Section 5 that our conjecture on gap facets holds for all the sequences that take two distinct values (in absolute value).

2 Preliminaries

The cut polytope.

Set $V := \{1, \ldots, n\}$. Let $\binom{n}{2}$ denote the set of unordered pairs ij with $1 \leq i < j \leq n$ (i.e., ij and ji are considered identical). Given a subset $S \subseteq V$, the set

$$\delta(S) := \{ij \in \tbinom{n}{2} : |S \cap \{i,j\}| = 1\}$$

is called the **cut** determined by S. Then, the polytope

$$\mathrm{CUT}_n := \mathrm{Conv}\{\chi^{\delta(S)} \mid S \subseteq V\},$$

which is defined as the convex hull of the incidence vectors of all cuts, is called the **cut polytope**. (For a set $A \subseteq \binom{n}{2}$, $\chi^A \in \{0,1\}^{\binom{n}{2}}$ denotes its incidence vector defined by $\chi^A_{ij} = 1$ if $ij \in A$ and $\chi^A_{ij} = 0$ if $ij \in \binom{n}{2} \setminus A$.) Given $v_0 \in \mathbb{R}$ and $v \in \mathbb{R}^{\binom{n}{2}}$, the inequality $v^T x \leq v_0$ is said to be **valid** for CUT_n if it is satisfied by all $x \in \mathrm{CUT}_n$ or, equivalently, by the incidence vectors of all cuts. A cut $\delta(S)$ for which equality $v^T \chi^{\delta(S)} = v_0$ holds is called a **root** of the inequality $v^T x \leq v_0$; we may also say that the set S itself defines a root of $v^T x \leq v_0$. The inequality $v^T x \leq v_0$ defines a **facet** of CUT_n if there exist $\binom{n}{2}$ roots whose incidence vectors are linearly independent.

LEMMA 2.1 *The inequality* (1.2) *is valid for the cut polytope* CUT_n. *A cut* $\delta(S)$ *is a root of* (1.2) *if and only if* $b(S) = \frac{\sigma-\gamma}{2}$ *or* $b(S) = \frac{\sigma+\gamma}{2}$.

PROOF. For $S \subseteq V$, we have: $Q_n(b)^T \chi^{\delta(S)} = b(S)(\sigma(b) - b(S))$, which is less than or equal to $\frac{\sigma(b)^2 - \gamma(b)^2}{4}$ by definition of the gap $\gamma(b)$. $\square$

Given a weight function $w \in \mathbb{R}^{\binom{n}{2}}$, the **max-cut problem** is the problem of finding a cut $\delta(S)$ whose weight $\sum_{ij\in\delta(S)} w_{ij}$ is maximum; it can be formulated as

$$\max(w^T x \mid x \in \mathrm{CUT}_n). \tag{2.2}$$

The max-cut problem is NP-hard [GJ79]. In fact, computing the gap of a sequence $b \in \mathbb{Z}^n$ can be formulated as an instance of the max-cut problem. Namely, set $w_{ij} := b_i b_j$ for all $ij \in \binom{n}{2}$. Then,

$$\gamma(b) \le \gamma \iff \max(w^T x \mid x \in \mathrm{CUT}_n) \ge \frac{\sigma(b)^2 - \gamma^2}{4}.$$

(This is actually the original method used by Karp for deriving the NP-hardness of the max-cut problem from the NP-completeness of the partition problem, using $\gamma = 0$ in the above argument.)

Root patterns.

Let $b = (b_1, \ldots, b_n) \in \mathbb{Z}^n$. We let $\sigma(b)$ denote the sum $\sum_{1\le i\le n} b_i$ and $\gamma(b)$ denote the gap of b, defined by (1.1). We also denote $\sigma(b)$ and $\gamma(b)$ by σ and γ, respectively, if there is no ambiguity.

Let $Q_n(b)$ denote the vector of $\mathbb{R}^{\binom{n}{2}}$ indexed by the pairs ij $(1 \le i < j \le n)$ and defined by

$$Q_n(b)_{ij} := b_i b_j \ \text{ for } 1 \le i < j \le n.$$

Hence, the inequality (1.2) reads:

$$Q_n(b)^T x \le \frac{\sigma(b)^2 - \gamma(b)^2}{4}.$$

It is convenient to look at the different values that are taken by the integers $b_1, \ldots, b_n$. Let k denote the number of distinct coefficients that enter in the sequence b and let $a_1, \ldots, a_k$ denote the distinct values taken by the entries of b. Then, the set V is partitioned into $V = V_1 \cup \ldots \cup V_k$, where $b_j = a_h$ for all $j \in V_h$, $h = 1, \ldots, k$. Let $m_h := |V_h|$ denote the multiplicity of entry a_h. Then, $n = m_1 + \ldots + m_k$ and $\sigma(b) = m_1 a_1 + \ldots + m_k a_k$. In other words, b is the sequence:

$$b = (\underbrace{a_1, \ldots, a_1}_{m_1}, \ldots, \underbrace{a_h, \ldots, a_h}_{m_h}, \ldots, \underbrace{a_k, \ldots, a_k}_{m_k}). \tag{2.3}$$

Given a subset $S \subseteq V$ and $r := (r_1, \ldots, r_k) \in \mathbb{N}_+^k$, we say that S has **pattern** $r = (r_1, \ldots, r_k)$ if $|S \cap V_h| = r_h$ for $h = 1, \ldots, k$. Set

$$K^* := \{h \in \{1, \ldots, k\} \mid m_h \ge 2\}.$$

Then, the inequality (1.2) can be rewritten as:

$$(2.4)\qquad \sum_{h\in K^*}(a_h)^2\Big(\sum_{i<j,\ i,j\in V_h} x_{ij}\Big)+\sum_{1\le h<h'\le k} a_h a_{h'}\Big(\sum_{i\in V_h,\ j\in V_{h'}} x_{ij}\Big)\le \frac{\sigma^2-\gamma^2}{4}.$$

We can clearly suppose that the integers $a_1,\ldots,a_k$ are relatively prime, i.e., that $a_1\wedge\ldots\wedge a_k=1$. (For two integers $a,b\ge 1$, $a\wedge b$ denotes their g.c.d.) Due to switching, as will be explained below, we can also assume without loss of generality that $a_1,\ldots,a_k\ge 1$.

Let $\delta(S)$ be a root of the inequality (2.4). As $\delta(S)$ is defined by any of the two subsets S and $V\setminus S$, we can always assume that we choose S in such a way that $b(S)=\frac{\sigma+\gamma}{2}$. let $r_h:=|S\cap V_h|$ for $h=1,\ldots,k$. Then, $\delta(S)$ is a root of (2.4) if and only if $b(S)=\frac{\sigma+\gamma}{2}$, i.e., if

$$(2.5)\qquad \sum_{1\le i\le k} a_i r_i=\frac{\sigma+\gamma}{2}\ \text{ or, equivalently, }\ \gamma=\sum_{1\le i\le k} a_i(2r_i-m_i).$$

Let $\mathcal{P}$ denote the set of possible patterns for the roots of (2.4), i.e., $\mathcal{P}$ consists of the sequences $r\in\mathbb{N}^k$ for which (2.5) holds. The members of $\mathcal{P}$ are called the **root patterns** of the inequality (2.4).

Switching.

Given an integer sequence $b\in\mathbb{Z}^n$ and $S\subseteq V$, we define another sequence $b'\in\mathbb{Z}^n$ by setting

$$b'_i=-b_i\ \text{ if } i\in S,\ \ b'_i=b_i\ \text{ if } i\in V\setminus S.$$

We say that b' is obtained from b by **switching** on S. It is easy to check that b and b' have the same gap.

LEMMA 2.6 *Both sequences b and b' have the same gap.* □

In the same way, we say that the inequality:

$$(2.7)\qquad Q_n(b')^T x\le \frac{\sigma(b')^2-\gamma(b')^2}{4}$$

is obtained from the inequality:

$$(2.8)\qquad Q_n(b)^T x\le \frac{\sigma(b)^2-\gamma(b)^2}{4}$$

by **switching** on S. Hence, each class of gap inequalities with a given gap γ is closed under switching, i.e., any switching of a gap inequality with gap γ is again a gap inequality with the same gap γ. It is not difficult to check that the gap inequalities with gap 0 are precisely the switchings of the negative type inequalities (i.e., the inequalities (1.2) for $\sigma=0$). In the same way, the gap inequalities with gap 1 are all the inequalities that can be obtained from the hypermetric inequalities (i.e., the inequalities (1.2) for $\sigma=1$) by switching. The following results can be found in [DL92]. They imply that we can suppose without loss of generality to deal with integer positive sequences.

Lemma 2.9 (i) *The inequality* (2.8) *defines a facet of* CUT_n *if and only if the inequality* (2.7) *does.*

(ii) *Let* $b \in \mathbb{Z}^n$ *and* $c := (b, 0) \in \mathbb{Z}^{n+1}$. *Then,* $\gamma(c) = \gamma(b) := \gamma$ *and* $\sigma(c) = \sigma(b) := \sigma$. *The inequality:* $Q_n(b)^T x \leq \frac{\sigma^2 - \gamma^2}{4}$ *defines a facet of* CUT_n *if and only if the inequality:* $Q_{n+1}(c)^T x \leq \frac{\sigma^2 - \gamma^2}{4}$ *defines a facet of* $\mathrm{CUT}_{n+1}^{\square}$. □

A positive semidefinite relaxation for the cut polytope.

Our interest in the gap inequalities is also motivated by the fact that they arise as a strenghtening of some positive semidefinite constraints by decreasing their right hand sides as much as possible. Here we give more details. Consider the set

$$\mathcal{J}_n := \{x \in \mathbb{R}^{\binom{n}{2}} \mid Q_n(b)^T x \leq \frac{\sigma(b)^2}{4} \text{ for all } b \in \mathbb{Z}^n\}.$$

Clearly, $\mathcal{J}_n$ is a convex body in $\mathbb{R}^{\binom{n}{2}}$ that contains the cut polytope CUT_n. Indeed, the inequalities defining $\mathcal{J}_n$ are obtained from the gap inequalities (1.2) by relaxing their right hand sides from $\frac{\sigma(b)^2 - \gamma(b)^2}{4}$ to $\frac{\sigma(b)^2}{4}$. In other words, if we let $\mathcal{G}_n$ denote the convex body in $\mathbb{R}^{\binom{n}{2}}$ which is defined by the gap inequalities (1.2) for all $b \in \mathbb{Z}^n$, then we have the following inclusions:

$$\mathrm{CUT}_n \subseteq \mathcal{G}_n \subseteq \mathcal{J}_n.$$

Even though $\mathcal{J}_n$ is a weaker relaxation of CUT_n than $\mathcal{G}_n$, it enjoys some nice properties that $\mathcal{G}_n$ does not have. An important property of $\mathcal{J}_n$ is that one can optimize over it in polynomial time. Namely, given $w \in \mathbb{R}^{\binom{n}{2}}$, the problem:

$$(2.10) \qquad \begin{array}{ll} \max & w^T x \\ \text{s.t.} & x \in \mathcal{J}_n \end{array}$$

can be solved (with arbitrary precision) in polynomial time (see, e.g., [GW94]). To see it, note first that the separation problem for $\mathcal{J}_n$: *Given* $x \in \mathbb{R}^{\binom{n}{2}}$, *decide whether* $x \in \mathcal{J}_n$ *and, if not, find* $b \in \mathbb{R}^n$ *such that* $Q_n(b)^T x > \frac{\sigma(b)^2}{4}$, can be solved in polynomial time. Indeed, for $x \in \mathbb{R}^{\binom{n}{2}}$, consider the $n \times n$ symmetric matrix X with zero diagonal and with ij-th entry x_{ij}; then, one can easily check that

$$x \in \mathcal{J}_n \iff \text{the matrix } J - 2X \text{ is positive semidefinite,}$$

where J denotes the all ones matrix. Now, using the ellipsoid method (see [GLS88]) this implies that the optimization problem (2.10) can be solved in polynomial time.

Goemans and Williamson [GW94] have shown that $\mathcal{J}_n$ provides a good approximation of CUT_n. More precisely, they show that

$$\frac{\max(w^T x \mid x \in \mathcal{J}_n)}{\max(w^T x \mid x \in \mathrm{CUT}_n)} \leq 1.131 \quad \text{for all } w \in \mathbb{R}_+^{\binom{n}{2}}.$$

In contrast, the optimization problem over the body $\mathcal{G}_n$ is probably a hard problem. Indeed, several facts indicate that the separation problem for the gap inequalities is quite likely to be hard. Some results of Avis and Grishukhin [AG93] show that the separation problem is already hard for the class of hypermetric inequalities. For instance, they show that the following problem is NP-hard: *Given $x \in \mathbb{R}^{\binom{n}{2}}$, decide if x satisfies all hypermetric inequalities and, if not, find $b \in \mathbb{Z}^n$ with $\sigma(b) = 1$ and minimum $\sum_{1 \leq i \leq n} |b_i|$ such that $Q_n(b)^T x > 0$.*

Note, however, that the separation problem for the negative type inequalities can be solved in polynomial time. Indeed, by the result of Schoenberg [Sch35, Sch38], x satisfies all the negative type inequalities if and only if the symmetric $(n-1) \times (n-1)$ matrix $(p_{ij})_{1 \leq i,j \leq n-1}$ is positive semidefinite, where

$$\begin{array}{ll} p_{ii} := x_{in} & \text{for } i = 1, \ldots, n-1, \\ p_{ij} := \frac{1}{2}(x_{in} + x_{jn} - x_{ij} & \text{for } 1 \leq i < j \leq n-1. \end{array}$$

3 Gap conditions

In this section, we present some results on the gap of a sequence $b \in \mathbb{Z}^n$. We start with an upper bound on the gap.

PROPOSITION 3.1 *Let $b \in \mathbb{Z}^n$. Then, $\gamma(b) \leq \max_{1 \leq i \leq n} |b_i|$.*

PROOF. We can suppose without loss of generality that $1 \leq b_1 \leq \ldots \leq b_n$. Recall that $\gamma(b) = \min |x^T b|$, where the minimum is taken over all ± 1-vectors x. We indicate a choice of x for which $|x^T b| \leq \max_i b_i$. For this, set $x_i := (-1)^{i-1}$ for $i = 1, \ldots, n$. Let $S_i := \sum_{1 \leq j \leq i} x_i$ for $i = 1, \ldots, n$. It can be easily checked that

$$0 \geq S_{2i} \geq -b_{2i}, \quad \text{and } 0 \leq S_{2i+1} \leq b_{2i+1}$$

for all i. This shows the result. □

We now suppose that b is the sequence from (2.3), i.e.,

$$b = (\underbrace{a_1, \ldots, a_1}_{m_1}, \ldots, \underbrace{a_h, \ldots, a_h}_{m_h}, \ldots, \underbrace{a_k, \ldots, a_k}_{m_k}),$$

where $a_1, \ldots, a_k$ are relatively prime integers. Let γ denote the gap of b and $\sigma := b_1 + \ldots + b_n$. We remind that $\mathcal{P}$ denotes the set of root patterns, i.e., the set of sequences $r \in \mathbb{N}^k$ such that $\sum_{1\le h\le k} a_h r_h = \frac{\sigma+\gamma}{2}$. As the integers $a_1, \ldots, a_k$ are relatively prime, there exist some integers $u_1, \ldots, u_k \in \mathbb{Z}$ for which the **Bezout identity**

$$(3.2) \qquad \sum_{1\le h\le k} a_h u_h = 1$$

holds. This identity is very useful. In some cases, a suitable choice of the Bezout parameters u_i's permits to conclude that the gap of b is 0 or 1. We present such cases below.

LEMMA 3.3 *Let $u_1, \ldots, u_k \in \mathbb{Z}$ satisfy the Bezout identity* (3.2). *Suppose that there is a root pattern $r \in \mathcal{P}$ satisfying*

$$(3.4) \qquad \begin{cases} r_h \ge u_h & \text{if } u_h > 0, \\ m_h - r_h \ge -u_h & \text{if } u_h < 0, \end{cases}$$

for each $h = 1, \ldots, k$. Then, the gap is equal to 0 *or* 1.

PROOF. Let $S \subseteq V$ realizing a root with pattern r. Set $S_h := V_h \cap S$ for $h = 1, \ldots, k$. Then, $|S_h| = r_h$. We define a new set $T \subseteq V$ as follows: $T = T_1 \cup \ldots \cup T_k$, where

$T_h = S_h$ minus a set of u_h points of S_h, if $u_h > 0$,
$T_h = S_h$ plus a set of $|u_h|$ points of $V_h \setminus S_h$, if $u_h < 0$ and
$T_h = S_h$, if $u_h = 0$.

Then, $b(T) - b(V\setminus T) = b(S) - b(V\setminus S) - 2\sum_{1\le h\le k} u_h a_h = \gamma - 2$. This implies that $\gamma \in \{0, 1\}$. Otherwise, if $\gamma \ge 2$, then we have found a set T whith $b(T) - b(V\setminus T) = \gamma - 2 < \gamma$, which contradicts the definition of the gap γ. □

In fact, the parameters u_i in the Bezout identity can be chosen with arbitrary signs.

LEMMA 3.5 *For each sign pattern $\varepsilon \in \{-1, 1\}^k$ distinct from $(1, \ldots, 1)$ and from $(-1, \ldots, -1)$, there exists $u^\varepsilon \in \mathbb{Z}^k$ satisfying*

$$\begin{cases} u_i^\varepsilon \varepsilon_i \ge 0 & \text{for all } i = 1, \ldots, k, \\ \sum_{1\le i\le k} u_i^\varepsilon a_i = 1. \end{cases}$$

PROOF. Let $u \in \mathbb{Z}^k$ be a solution for the Bezout identity $\sum_{1\le i\le k} u_i a_i = 1$. Then, $u + x$ is another solution if $x \in \mathbb{Z}^k$ satisfies $\sum_{1\le i\le k} x_i a_i = 0$. The result now follows by taking for x a suitable combination of the vectors $(-a_i, 0, \ldots, 0, a_1, 0, \ldots, 0)$ (where a_1 stands in the i-th position) for $i = 2, \ldots, k$. □

Corollary 3.6 *Let u^ε be defined as in Lemma 3.5. Suppose that, for every $i = 1, \ldots, k$, $m_i \geq 2 \max_{\varepsilon \in \{1,-1\}^k, \varepsilon \neq (1,\ldots,1),(-1,\ldots,-1)} |u_i^\varepsilon|$. Then, b has gap $\gamma(b) \leq 1$.*

Proof. Let $S \subseteq V$ be a root of (1.2) with pattern r, i.e., $\gamma = \sum_{1 \leq i \leq k}(2r_i - m_i)a_i$. Then, $r_i > \frac{m_i}{2}$ for some i (else, $2r_i - m_i \leq 0$ for all i, implying $\gamma \leq 0$, a contradiction). Moreover, $r_j \leq \frac{m_j}{2}$ for some j. This can be seen as follows. For $i = 1, \ldots, k$, set

$$T_i := S_1 \cup \ldots \cup S_{i-1} \cup (V_i \setminus S_i) \cup S_{i+1} \cup \ldots \cup S_k,$$

where $S_j = V_j \cap S$ for all j. Then, $\sum_{1 \leq i \leq k} b(T_i) - b(V \setminus T_i) = (k-2)\gamma \geq 0$. We can suppose, for instance, that $b(T_1) - b(V \setminus T_1) \geq 0$. This implies that $b(T_1) - b(V \setminus T_1) \geq \gamma$, i.e., $2a_1(2r_1 - m_1) \leq 0$ and, thus, $r_1 \leq \frac{m_1}{2}$. Set

$$I := \{i \in \{1, \ldots, k\} \mid r_i \leq \frac{m_i}{2}\},\ J := \{i \in \{1, \ldots, k\} \mid r_i > \frac{m_i}{2}\}.$$

So, $I, J \neq \emptyset$. Let $\varepsilon \in \{-1, 1\}^k$ be defined by $\varepsilon_i = -1$ if $i \in I$ and $\varepsilon_i = 1$ if $i \in J$. Let u^ε be defined as in Lemma 3.5. Hence, $u_i^\varepsilon \varepsilon_i \geq 0$ for all i. We check that the assumptions of Lemma 3.3 hold. Indeed, if $u_i^\varepsilon > 0$ then $r_i > \frac{m_i}{2}$ as $i \in J$ and, thus, $r_i \geq u_i^\varepsilon$ as $u_i^\varepsilon \leq \frac{m_i}{2}$ by assumption. On the other hand, if $u_i^\varepsilon < 0$ then $r_i \leq \frac{m_i}{2}$ as $i \in I$; hence, $r_i - u_i^\varepsilon \leq \frac{m_i}{2} + \frac{m_i}{2} = m_i$, i.e., $m_i - r_i \geq |u_i^\varepsilon|$. Therefore, by Lemma 3.3, we can conclude that the gap is 0 or 1. $\square$

4 Facet conditions

In this section, we study when the gap inequality (2.4) defines a facet of the cut polytope. We give necessary and sufficient conditions for the inequality (2.4) to be facet defining. These conditions are in terms of root patterns; see Theorem 4.4 and Propositions 4.14 and 4.19. Our characterization presents the interesting feature that it is expressed in terms of conditions on the root patterns, which live in the n-space, while the facet property concerns the $\binom{n}{2}$-space.

4.1 Facet characterization

We remind that K^* denotes the set of indices $h = 1, \ldots, k$ for which $m_h \geq 2$. Set

$$J = \{hh' \ : 1 \leq h < h' \leq k \text{ or } h = h' \in K^*\}.$$

Hence, $|J| = \binom{k}{2} + |K^*|$. Based on the family $\mathcal{P}$ of root patterns, we introduce a $|\mathcal{P}| \times |J|$ matrix $M_\mathcal{P}$. The rows of $M_\mathcal{P}$ correspond to the root patterns $r \in \mathcal{P}$, and its columns to the pairs $hh' \in J$. The entry of $M_\mathcal{P} = (m_{r,hh'})$ in the row corresponding to r and in the column hh' is given by

$$m_{r,hh'} := \begin{cases} r_h(m_{h'} - r_{h'}) + r_{h'}(m_h - r_h) & \text{if } h \neq h' \\ r_h(m_h - r_h) & \text{if } h = h' \end{cases}$$

We now formulate some necessary conditions for the inequality (2.4) to be facet defining.

LEMMA 4.1 *If the inequality* (2.4) *defines a facet of the cut polytope, then*

$$\text{rank } M_{\mathcal{P}} = \binom{k}{2} + |K^*|.$$

PROOF. Set $\alpha_{hh'} := a_h a_{h'}$ for all $hh' \in J$. Then, by construction of the matrix $M_{\mathcal{P}}$, the vector $y = \alpha$ satisfies the system:

$$M_{\mathcal{P}} y = \frac{\sigma^2 - \gamma^2}{4}. \tag{4.2}$$

Assume that rank $M_{\mathcal{P}} < |J|$. Then (4.2) has another solution $\beta \neq \alpha$, and the following equality:

$$\sum_{hh' \in J} \beta_{hh'} \Big(\sum_{i \in V_h,\ j \in V_{h'}} x_{ij} \Big) = \frac{\sigma^2 - \gamma^2}{4}.$$

is satisfied by all roots of (2.4). This proves that (2.4) is not facet defining. □

Let $G = (V, E)$ be a graph and let $\mathcal{S}$ be a collection of subsets of V. Set

$$\mathcal{F}_{\mathcal{S}}^G := \{\chi^{\delta_G(S)} \mid S \in \mathcal{S}\},$$

where $\chi^{\delta_G(S)}$ is the characteristic vector of the cut determined by the set S in the graph G. We say that $\mathcal{F}_{\mathcal{S}}^G$ is **full dimensional** if $\mathcal{F}_{\mathcal{S}}^G$ spans the whole space $\mathbb{R}^E$.
We will consider here the following cases:
- For $h \in K^*$, G is the complete graph on V_h and $\mathcal{S} = \{S \subseteq V_h : |S| = r_h \text{ for some } r \in \mathcal{P}\}$; then, $\mathcal{F}_{\mathcal{S}}^G$ is denoted as $\mathcal{F}_{h,h}$.
- For $1 \leq h < h' \leq k$, G is the complete bipartite graph with node bipartition $V_h \cup V_{h'}$ and $\mathcal{S} = \{S \subseteq V_h \cup V_{h'} : |S \cap V_h| = r_h \text{ and } |S \cap V_{h'}| = r_{h'} \text{ for some } r \in \mathcal{P}\}$; then, $\mathcal{F}_{\mathcal{S}}^G$ is denoted as $\mathcal{F}_{h,h'}$.

LEMMA 4.3 *Assume that the inequality* (2.4) *is facet defining. Then, the set* $\mathcal{F}_{h,h'}$ *is full dimensional for each* $h = h' \in K^*$ *and for each* $1 \leq h < h' \leq k$.

PROOF. If the inequality (2.4) is facet defining, then its set X of roots has full dimension $\binom{n}{2}$. Therefore, the set $\{(x_{ij})_{i\in V_h, j\in V_{h'}} \mid x \in X\}$ is a subset of $\mathcal{F}_{h,h'}$ of full dimension. □

In fact, as the next result shows, the conditions from Lemmas 4.1 and 4.3 are already sufficient for characterizing facets.

THEOREM 4.4 *The inequality* (2.4) *defines a facet of the cut polytope if and only if the following conditions* (i) *and* (ii) *hold:*

(i) rank $M_{\mathcal{P}} = \binom{k}{2} + |K^*|$.

(ii) *The set $\mathcal{F}_{h,h'}$ is full dimensional for every $h = h' \in K^*$ and every $1 \le h < h' \le k$.*

PROOF. We suppose that the conditions (i) and (ii) hold. We show that the inequality (2.4) defines a facet of the cut polytope. For this, let $v \in \mathbb{R}^{\binom{n}{2}}$ and $v_0 \in \mathbb{R}$ such that all roots of (2.4) satisfy the equation:

$$v^T x = v_0. \tag{4.5}$$

We show that the equations (4.5) and (2.4) define the same hyperplane.
Set $\beta_{hh'} := \sum_{i\in V_h, j\in V_{h'}} v_{ij}$ for every $hh' \in J$, and $\beta = (\beta_{hh'})$. For each root pattern $r \in \mathcal{P}$, let X_r denote the set of roots of (2.4) with pattern r, i.e.,

$$X_r = \{\chi^{\delta(S)} \mid S \subseteq V \text{ with } |S \cap V_h| = r_h \text{ for } h = 1, \ldots, k\},$$

and set

$$\sigma^{(r)} = \sum_{x\in X_r} x.$$

The components of $\sigma^{(r)}$ are given by

$$\sigma_{ij}^{(r)} = |X_r| \cdot \tfrac{2r_h(m_h - r_h)}{m_h(m_h-1)} \qquad \text{for } i < j, i, j \in V_h,\ h = 1, \ldots, k,$$

$$\sigma_{ij}^{(r)} = |X_r| \cdot \tfrac{r_h(m_{h'} - r_{h'}) + r_{h'}(m_h - r_h)}{m_h m_{h'}} \qquad \text{for } i \in V_h, j \in V_{h'},\ 1 \le h < h' \le k,$$

where $|X_r| = \prod_{1\le h\le k} \binom{m_h}{r_h}$. Summing (4.5) over all roots $x \in X_r$, we obtain: $\sum_{x\in X_r} v^T x = v^T \sigma^{(r)} = |X_r| v_0$. Define the vector $\beta' \in \mathbb{R}^J$ by setting

$$\beta'_{hh} := \tfrac{2\beta_{hh}}{m_h(m_h-1)} \qquad \text{for } h \in K^*,$$

$$\beta'_{hh'} := \tfrac{\beta_{hh'}}{m_h m_{h'}} \qquad \text{for } 1 \le h < h' \le k.$$

Hence, we have

$$(4.6) \qquad M_{\mathcal{P}}\beta' = v_0 e,$$

where e denotes the all ones vector.

Consider a pair $h_0h_0' \in J$ and a root $\overline{x}$ of pattern r, i.e., $\overline{x} \in X_r$. We show that the quantity:

$$\sum_{i \in V_{h_0}, j \in V_{h_0'}} v_{ij}\overline{x}_{ij}$$

is a constant depending only on the root pattern r (and not on the choice of $\overline{x} \in X_r$). For this, let Y denote the subset of X_r defined by $Y := \{x \in X_r \mid x_{ij} = \overline{x}_{ij}$ for $i \in V_{h_0}, j \in V_{h_0'}\}$. Then,

$$\begin{aligned}
\sum_{x \in Y} v^T x &= \sum_{x \in Y} \Big(\sum_{i \in V_{h_0}, j \in V_{h_0'}} v_{ij}x_{ij} + \sum_{hh' \in J \setminus h_0h_0'} \sum_{i \in V_h, j \in V_{h'}} v_{ij}x_{ij} \Big) \\
&= |Y| \cdot \sum_{i \in V_{h_0}, j \in V_{h_0'}} v_{ij}\overline{x}_{ij} + \sum_{hh' \in J \setminus h_0h_0'} \sum_{i \in V_h, j \in V_{h'}} v_{ij} \Big(\sum_{x \in Y} x_{ij} \Big) \\
&= |Y| \cdot \sum_{i \in V_{h_0}, j \in V_{h_0'}} v_{ij}\overline{x}_{ij} + \sum_{hh' \in J \setminus h_0h_0'} c_{hh'} \Big(\sum_{i \in V_h, j \in V_{h'}} v_{ij} \Big) \\
&= |Y| \cdot \sum_{i \in V_{h_0}, j \in V_{h_0'}} v_{ij}\overline{x}_{ij} + \sum_{hh' \in J \setminus h_0h_0'} c_{hh'}\beta_{hh'}.
\end{aligned}$$

Here we used the fact that $c_{hh'} := \sum_{x \in Y} x_{ij}$ is a constant independent of $i \in V_h$ and $j \in V_{h'}$ for every fixed pair $hh' \in J \setminus h_0h_0'$. On the other hand, $\sum_{x \in Y} v^T x = v_0|Y|$ which implies

$$(4.7) \qquad v_0|Y| = |Y| \cdot \sum_{i \in V_{h_0}, j \in V_{h_0'}} v_{ij}\overline{x}_{ij} + \sum_{hh' \in J \setminus h_0h_0'} c_{hh'}\beta_{hh'}.$$

This shows that the quantity

$$(4.8) \qquad w^{(r)}_{h_0h_0'} := \sum_{i \in V_{h_0}, j \in V_{h_0'}} v_{ij}\overline{x}_{ij}$$

is a constant depending only on the root pattern r. Summing over $\overline{x} \in X_r$, we obtain that

$$\begin{aligned}
w^{(r)}_{hh} &= \beta_{hh} \frac{r_h(m_h - r_h)}{\binom{m_h}{2}} && \text{for } h \in K^*, \\
w^{(r)}_{hh'} &= \beta_{hh'} \frac{r_h(m_{h'} - r_{h'}) + r_{h'}(m_h - r_h)}{m_h m_{h'}} && \text{for } 1 \le h < h' \le k.
\end{aligned}$$

Suppose first that $v_0 = 0$. Then, we deduce from (4.6) that $\beta' = 0$, as the matrix $M_{\mathcal{P}}$ has full column rank. Therefore, $\beta = 0$. Relation (4.7) implies that

$$\sum_{i \in V_{h_0}, j \in V_{h'_0}} v_{ij} \overline{x}_{ij} = 0$$

for each root $\overline{x}$ and each $h_0 h'_0 \in J$. As each family $\mathcal{F}_{h_0,h'_0}$ is full dimensional, this implies that $v = 0$, a contradiction.

Therefore, $v_0 \neq 0$. We can suppose without loss of generality that $v_0 = \frac{\sigma^2 - \gamma^2}{4}$. As $M_{\mathcal{P}}$ has full column rank, we deduce from relation (4.6) that

$$\beta'_{hh'} = a_h a_{h'} \text{ for all } hh' \in J.$$

From (4.8),

$$\sum_{i \in V_{h_0}, j \in V_{h'_0}} v_{ij} x_{ij} = w^{(r)}_{h_0 h'_0}$$

for all roots x with pattern r. Using the above formulas for $w^{(r)}_{h_0 h'_0}$ and the fact that each family $\mathcal{F}_{h_0,h'_0}$ is full dimensional, we deduce that v_{ij} (for $i \in V_{h_0}, j \in V_{h'_0}$) is a constant depending only on h_0, h'_0. This shows that $v_{ij} = a_{h_0} a_{h'_0}$ for all $i \in V_{h_0}, j \in V_{h'_0}$. □

We show in the next subsection how the conditions on the full dimensionality of the cut families $\mathcal{F}_{h,h'}$ can be reformulated as simple conditions on the set $\mathcal{P}$ of root patterns; see Propositions 4.14 and 4.19.

4.2 Linear dependencies of uniform cuts

Our objective in this section is to give a reformulation of Theorem 4.4 which uses only simple conditions on the root patterns of the inequality (2.4). For this, we need to formulate conditions for the full dimensionality of the cut families $\mathcal{F}_{h,h'}$. Such conditions are given in Propositions 4.14 and 4.19.

Given $h \in K^*$, we recall that the cut family $\mathcal{F}_{h,h}$ consists of the incidence vectors of the cuts $\delta_G(S)$, where G is the complete graph on V_h and $S \subseteq V_h$ with $|S| = r_h$ for some $r \in \mathcal{P}$. Hence, $\mathcal{F}_{h,h}$ is a union of several families of uniform cuts. In fact, as we will see below, the full dimensionality of $\mathcal{F}_{h,h}$ can be checked by using at most two different set sizes for the cuts in $\mathcal{F}_{h,h}$.

We start with the problem of determining when the family of all uniform cuts of a given size is full dimensional.
Let m and r be integers satisfying $m \geq 2,\ 1 \leq r \leq m-1$. Let

$$\mathcal{C}_r := \{\chi^{\delta(S)} \mid S \subset \{1, \ldots, m\},\ |S| = r\} \subset \mathbb{R}^{\binom{m}{2}}$$

be the collection of characteristic vectors of the cuts $\delta(S)$ in K_m satisfying $|S| = r$. We are interested in determining when does $\mathcal{C}_r$ span the whole space $\mathbb{R}^{\binom{m}{2}}$; we call $\mathcal{C}_r$ **full dimensional** in that case. Clearly, the set $\mathcal{C}_r$ is full dimensional if and only if all unit vectors belong to the span of $\mathcal{C}_r$, i.e., if, for any fixed pair $i_0,\ j_0 \in V = \{1, 2, \ldots, m\},\ i_0 \neq j_0$, the vector $e(i_0, j_0) \in \mathbb{R}^{\binom{m}{2}}$ defined by

$$e(i_0, j_0)_{ij} = \begin{cases} 1 & i = i_0,\ j = j_0, \\ 0 & \text{otherwise}, \end{cases}$$

can be expressed as a linear combination of the form:

$$e(i_0, j_0) = \sum_{S \in \mathcal{S}} \alpha_S \chi^{\delta(S)}$$

where $\mathcal{S} = \{S : |S| = r,\ S \subseteq V\}$. If such a linear combination $\alpha = (\alpha_S)$ exists then, due to the underlying symmetries, α can be assumed to have a particular form. Namely, we can suppose that there are three coefficients $\beta_0, \beta_1, \beta_2$ so that $\alpha_S = \beta_k$ for all $S \in \mathcal{S}_k,\ k = 0, 1, 2$, where $\mathcal{S}_k$ are defined by

$$\mathcal{S}_k \ := \{S \in \mathcal{S} \ : \ |S \cap \{i_0, j_0\}| = k\}$$

for $k = 0, 1, 2$. Equivalently, we may ask whether the unit vector $e(i_0, j_0)$ can be expressed as a linear combination of the three vector $\sigma^k,\ k = 0, 1, 2$, given by

$$\sigma^k \ := \sum_{S \in \mathcal{S}_k} \chi^{\delta(S)}.$$

Therefore, the question whether the family $\mathcal{C}_r$ is full dimensional is equivalent to the question whether there exists a solution $\beta = (\beta_0, \beta_1, \beta_2)$ to the linear system:

$$\beta_0 \sigma^0 + \beta_1 \sigma^1 + \beta_2 \sigma^2 = e(i_0, j_0). \tag{4.9}$$

This question is answered by Lemma 4.12. In fact, it can be further reduced in the following way. For this, it is convenient to introduce some auxiliary matrices associated with the cuts of the graph K_m.
For every integer $r,\ 0 \le r \le m$, we introduce a 3×3 matrix $A_r = (a_{ij}),\ i, j = 0, 1, 2$, as follows. Fix the edge $\overline{e} := i_0 j_0$ of the complete graph K_m, and partition its edge set into three classes $E_0,\ E_1$ and E_2 according to the intersection with $\overline{e}$:

$$E_i := \{e \ : \ |e \cap \overline{e}| = i\} \qquad i = 0, 1, 2.$$

(Thus, $E_2 = \{\overline{e}\}$). Now, we define the entries a_{ij} of the matrix A_r by

$$a_{ij} = \begin{cases} |E_i \cap \delta(S)| & \text{if there is a subset } S \subset \{1, \ldots, n\}, |S| = r,\ |S \cap \overline{e}| = j \\ 0 & \text{if such a set } S \text{ does not exist.} \end{cases} \tag{4.10}$$

Observe that the value of a_{ij} is independent of a particular choice of a subset S. Clearly, the system (4.9) is solvable if and only if the system:

$$(4.11) \qquad A_r x = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

is solvable. In other words, the uniform cut family $\mathcal{C}_r$ is full dimensional if and only if the system (4.11) is solvable. In the next result, we characterize the values of r for which $\mathcal{C}_r$ is full dimensional. We also give some values of r, r' for which the family $\mathcal{C}_r \cup \mathcal{C}_{r'}$ is full dimensional.

LEMMA 4.12 *Let A_r denote the matrix defined by* (4.10).

(i) *Assume $m \geq 4$. Then, the linear system* (4.11) *is solvable if and only if $r \neq 0, 1, \frac{1}{2}m, m-1, m$.*

(ii) *Assume $m = 2, 3$. Then,* (4.11) *is solvable if and only if $r = 1,\ m-1$.*

(iii) *Assume $m \geq 2$ and m is even. Then, the linear system:*

$$(4.13) \quad A_{\frac{m}{2}} x + A_1 y = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

is solvable.

PROOF. (i) Set $s := m - r$. For $r \neq 0, 1, m-1, m$, the matrix A_r reads

$$A_r = \begin{pmatrix} r(s-2) & (r-1)(s-1) & (r-2)s \\ 2r & r+s-2 & 2s \\ 0 & 1 & 0 \end{pmatrix}$$

For $r \neq \frac{m}{2}$, the system (4.11) has the solution:

$$\begin{aligned} x_0 &= \tfrac{1}{r-s}\left(1 - \tfrac{m}{2} + \tfrac{1}{r} - s\right) \\ x_1 &= 1 \\ x_2 &= \tfrac{1}{s-r}\left(1 - \tfrac{m}{2} + \tfrac{1}{s} - r\right). \end{aligned}$$

Assume $r = \frac{m}{2}$. Hence $s = r = \frac{m}{2}$. Summing all three equations of (4.11) together yields

$$r^2(x_0 + x_1 + x_2) = 1$$

Summing the second equation with a double of the third one yields

$$r(x_0 + x_1 + x_2) = 1.$$

Hence (4.11) is not solvable for $r = s = \frac{m}{2}$ and $m \geq 4$.
Let $r = 1$. For $m \geq 3$ we have:

$$A_1 = \begin{pmatrix} m-3 & 0 & 0 \\ 2 & m-2 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and, hence, (4.11) is solvable if and only if $m = 3$.
Assume $r = 1$ and $m = 2$. Then

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and (4.11) is solvable.
The case $r = m - 1$ is analogous to $r = 1$.
If $r = 0$ or $r = m$, then $A_r = 0$ and, hence, (4.11) is not solvable.
The previous two arguments prove the validity of (ii).

We now check (iii). If $m = 2$, the validity of (iii) follows from (ii). Hence, assume $m \geq 4$ and $r = m/2$. Modify (4.13) as follows. Add the sum of the last two equations to the first equation, and add the multiple two of the third equation to the second one. Thus, (4.13) is equivalent to the system:

$$\begin{pmatrix} r^2 & r^2 & r^2 & 2r-1 & 2r-1 & 0 \\ 2r & 2r & 2r & 2 & 2r & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ y_0 \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

Setting $x_1 = x_2 = y_2 = 0$, we obtain the linear system:

$$\begin{pmatrix} r^2 & 2r-1 & 2r-1 \\ 2r & 2 & 2r \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

which is solvable, since the determinant of its matrix is $2r(1-r) \neq 0$, as $r > 1$. $\square$

We can now characterize when the cut family $\mathcal{F}_{h,h}$ is full dimensional, in terms of the set $\mathcal{P}$ of root patterns. Let

$$\mathcal{P}_h := \{\rho \mid \exists r \in \mathcal{P} \text{ such that } r_h = \rho\}$$

denote the projection of $\mathcal{P}$ on the h-th coordinate.

PROPOSITION 4.14 *Let $h \in K^*$. If $m_h = 2, 3$, then the family $\mathcal{F}_{h,h}$ is full dimensional if and only if there exists a root pattern $r \in \mathcal{P}$ such that $r_h \in \{1, m_h - 1\}$. If $m_h \geq 4$, then the family $\mathcal{F}_{h,h}$ is full dimensional if and only if one of the following conditions holds:*

(i) *There exists a root pattern $r \in \mathcal{P}$ such that $r_h \notin \{0, 1, \frac{m_h}{2}, m_h - 1, m_h\}$.*

(ii) *$\mathcal{P}_h \subseteq \{0, 1, \frac{m_h}{2}, m_h - 1, m_h\}$; m_h is even; $\frac{m_h}{2} \in \mathcal{P}_h$; at least one of 1 and $m_h - 1$ belongs to $\mathcal{P}_h$.*

PROOF. Suppose first that $m_h = 2, 3$. If $r_h \in \{0, m_h\}$ for all $r \in \mathcal{P}$ then $\mathcal{F}_{h,h}$ is reduced to the zero vector. On the other hand, if $r_h \in \{1, m_h - 1\}$ for some $r \in \mathcal{P}$ then $\mathcal{F}_{h,h}$ is full dimensional by Lemma 4.12 (ii). Suppose now that $m_h \geq 4$. If $r_h \notin \{0, 1, \frac{m_h}{2}, m_h - 1, m_h\}$ for some $r \in \mathcal{P}$, then $\mathcal{F}_{h,h}$ is full dimensional by Lemma 4.12 (i). Otherwise, $\mathcal{P}_h \subseteq \{0, 1, \frac{m_h}{2}, m_h - 1, m_h\}$. If $\mathcal{F}_{h,h}$ is full dimensional then $\frac{m_h}{2} \in \mathcal{P}_h$ and $\mathcal{P}_h \cap \{1, m_h - 1\} \neq \emptyset$ (by Lemma 4.12 (i)); if the latter conditions hold then $\mathcal{F}_{h,h}$ is full dimensional by Lemma 4.12 (iii). □

We now turn to the study of the cut families $\mathcal{F}_{h,h'}$ where $1 \leq h < h' \leq k$. Note that the family $\mathcal{F}_{h,h'}$ is a union of uniform cut families in the complete bipartite graph $G = K_{m_h, m_{h'}}$ with node bipartition $V_h \cup V_{h'}$. We will see that the full dimensionality of $\mathcal{F}_{h,h'}$ can be checked by looking at two set sizes at most.

As in the case treated above, we first study the case when the set of uniform cuts of a given size in a complete bipartite graph is full dimensional. Again, due to symmetries, this problem can be formulated as follows.

Let $m_1, m_2 \geq 1$ be fixed. Consider the complete bipartite graph K_{m_1,m_2} with vertex set $V = V_1 \cup V_2$, $|V_i| = m_i$, $i = 1, 2$, and choose a pair of vertices $i_0 \in V_1$ and $j_0 \in V_2$. Partition the edges of K_{m_1,m_2} into four sets as follows:

$$\begin{aligned}
E_1 &:= \{i_0 j_0\}, \\
E_2 &:= \{i_0 j \mid j \in V_2 \backslash j_0\}, \\
E_3 &:= \{i j_0 \mid i \in V_1 \backslash i_0\}, \\
E_4 &:= \{ij \mid i \in V_1 \backslash i_0,\ j \in V_2 \backslash i_0\}.
\end{aligned}$$

For any pair of integers r_1 and r_2, $0 \leq r_i \leq m_i$, $i = 1, 2$, let

$$\mathcal{S}_{r_1,r_2} := \{S \subseteq V_1 \cup V_2 \ :\ |S \cap V_i| = r_i,\ i = 1, 2\}.$$

We partition the set system $\mathcal{S}_{r_1,r_2}$ into four classes $\mathcal{S}_i$, $i = 1, 2, 3, 4$, as follows:

$$\begin{aligned}
\mathcal{S}_1 &:= \{S \in \mathcal{S}_{r_1,r_2} \mid i_0, j_0 \in S\}, \\
\mathcal{S}_2 &:= \{S \in \mathcal{S}_{r_1,r_2} \mid i_0 \in S,\ j_0 \notin S\}, \\
\mathcal{S}_3 &:= \{S \in \mathcal{S}_{r_1,r_2} \mid i_0 \notin S,\ j_0 \in S\}, \\
\mathcal{S}_4 &:= \{S \in \mathcal{S}_{r_1,r_2} \mid i_0, j_0 \notin S\}.
\end{aligned}$$

We introduce a matrix $B_{r_1,r_2} = (b_{k,\ell}),\ k,\ell = 1,2,3,4$, by setting

$$b_{k,\ell} := \begin{cases} |E_k \cap \delta(S)| & \textit{for } S \in \mathcal{S}_\ell \\ 0 & \textit{if } \mathcal{S}_\ell = \emptyset. \end{cases}$$

Clearly, the value of $b_{k,\ell}$ is independent of a particular choice of $S \in \mathcal{S}_\ell$ for any $\ell = 1,2,3,4$. We are interested in the solvability of the linear system:

$$B_{r_1,r_2}x = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{4.15}$$

Clearly, the system (4.15) is solvable if and only if the family

$$\mathcal{C}_{r_1,r_2} := \{\chi^{\delta_G(S)} \mid S \in \mathcal{S}_{r_1,r_2}\}$$

is full dimensional (where $G = K_{m_1,m_2}$). We now characterize the values of (r_1, r_2) for which the family $\mathcal{C}_{r_1,r_2}$ is full dimensional. We also give some values of (r_1, r_2) and (r_1', r_2') for which the union $\mathcal{C}_{r_1,r_2} \cup \mathcal{C}_{r_1',r_2'}$ is full dimensional.

LEMMA 4.16 (i) *Let $m_1 = 1$ and $m_2 \geq 2$. The system (4.15) is solvable for $r_1 \in \{0,1\}$ and $r_2 \neq 0, m_2$.*

(ii) *Let $m_1, m_2 \geq 2$. Then, (4.15) is solvable if and only if $r_i \neq 0, \frac{1}{2}m_i, m_i$ for $i = 1,2$.*

(iii) *Let $m_1, m_2 \geq 2$. The system:*

$$B_{m_1,r_2}\ x + B_{\frac{m_1}{2},r_2'}\ y = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

is solvable for $r_2 \neq 0, m_2$ and $r_2' \neq \frac{m_2}{2}$.

(iv) *Let $m_1, m_2 \geq 2$. The system:*

$$B_{\frac{m_1}{2},r_2}x + B_{r_1,\frac{m_2}{2}}y = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

is solvable for $r_i \neq 0,\ \frac{m_i}{2},\ m_i$, for $i = 1,2$.

Before giving the proof, we introduce another result that will also be needed. Let H_0, H_1, H_2 denote the hyperplanes in $\mathbb{R}^4$ that are defined by

$$H_0 = \{z \in \mathbb{R}^4 \mid z_1 + z_2 + z_3 + z_4 = -m_1m_2z_1 + m_1(z_1 + z_2) + m_2(z_1 + z_3)\},$$
$$H_1 := \{z \in \mathbb{R}^4 \mid z_1 + z_2 + z_3 + z_4 = m_2(z_1 + z_3)\},$$
$$H_2 := \{z \in \mathbb{R}^4 \mid z_1 + z_2 + z_3 + z_4 = m_1(z_1 + z_2)\}.$$

Observe that the vector $(1,0,0,0)^T$ does not belong to any of the hyperplanes H_0, H_1, or H_2 if $m_1, m_2 \geq 2$. For a matrix B, the **range** of B is the set consisting of the vectors Bx for $x \in \mathbb{R}^4$ (if B has four columns).

LEMMA 4.17 *Let* $m_1, m_2 \geq 2$.

(i) *For every* r_1*, the range of each of the matrices* B_{r_1,m_2} *and* $B_{r_1,0}$ *is contained in both hyperplanes* H_0 *and* H_1.

(ii) *For every* r_2*, the range of each of the matrices* B_{m_1,r_2} *and* B_{0,r_2} *is contained in both hyperplanes* H_0 *and* H_2.

(iii) *For every* r_1*, the range of the matrix* $B_{r_1,\frac{m_2}{2}}$ *is contained in the hyperplane* H_2.

(iv) *For every* r_2*, the range of the matrix* $B_{\frac{m_1}{2},r_2}$ *is contained in the hyperplane* H_1.

The following notation will be useful for the proofs of Lemmas 4.16 and 4.17. For a 4×4 matrix B with rows u_1, u_2, u_3, u_4, we let B' denote the 4×4 matrix whose rows are the vectors $u_1,\ u_1+u_2,\ u_1+u_3$, and $u_1+u_2+u_3+u_4$. So, B'_{r_1,r_2} is the transform of B_{r_1,r_2} defined in this way. Obviously, the system (4.15) is solvable if and only if the system:

$$B'_{r_1,r_2} x = e \tag{4.18}$$

is solvable, where $e := (1,1,1,1)^T$.

We collect below a list of matrices, which show all the possible forms for the matrices B'_{r_1,r_2} in the case when $m_1, m_2 \geq 2$:

$$B'_{r_1,r_2} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ s_2 & s_2 & r_2 & r_2 \\ s_1 & r_1 & s_1 & r_1 \\ r_1 s_2 + s_1 r_2 & r_1 s_2 + s_1 r_2 & r_1 s_2 + s_1 r_2 & r_1 s_2 + s_1 r_2 \end{pmatrix}$$

for $r_1 \neq 0, m_1,\ r_2 \neq 0, m_2$, and setting $s_1 := m_1 - r_1,\ s_2 = m_2 - r_2$;

$$B'_{m_1,r_2} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ s_2 & s_2 & 0 & 0 \\ 0 & m_1 & 0 & 0 \\ m_1 s_2 & m_1 s_2 & 0 & 0 \end{pmatrix}, \quad B'_{r_1,m_2} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & m_2 & 0 \\ s_1 & 0 & s_1 & 0 \\ s_1 m_2 & 0 & s_1 m_2 & 0 \end{pmatrix},$$

for $r_2 \neq 0, m_2,\ s_2 := m_2 - r_2$, and $r_1 \neq 0, m_1,\ s_1 := m_1 - r_1$;

$$B'_{0,r_2} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & r_2 & r_2 \\ 0 & 0 & m_1 & 0 \\ 0 & 0 & r_2 m_1 & r_2 m_1 \end{pmatrix}, \quad B'_{r_1,0} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & r_1 & 0 & r_1 \\ 0 & r_1 m_2 & 0 & r_1 m_2 \end{pmatrix},$$

for $r_2 \neq 0, m_2$, and $r_1 \neq 0, m_1$;

$$B'_{0,m_2} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & m_2 & 0 \\ 0 & 0 & m_1 & 0 \\ 0 & 0 & m_1 m_2 & 0 \end{pmatrix}, \quad B'_{m_1,0} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & m_1 & 0 & 0 \\ 0 & m_1 m_2 & 0 & 0 \end{pmatrix}.$$

PROOF OF LEMMA 4.17. By construction of the matrix B'_{r_1,r_2}, we have: $z = B_{r_1,r_2}x$ if and only if $B'_{r_1,r_2}x = z'$, where z' is the vector $(z_1, z_1 + z_2, z_1 + z_3, z_1 + z_2 + z_3 + z_4)^T$. The claims from Lemma 4.17 can be easily verified by inspection of the matrices B'_{r_1,r_2}. □

PROOF OF LEMMA 4.16. (i) Let $r_1 = 1$ and $r_2 \neq 0, m_2$. then, the system (4.15) reads:

$$\begin{aligned} & x_2 = 1 \\ & (m_2 - r_2)x_1 + r_2 x_2 = 0. \end{aligned}$$

Hence, it is solvable. The case $r_1 = 0$ is analogous.
(ii) Suppose first that the system (4.15) is solvable. Hence, the vector $(1,0,0,0)^T$ belongs to the range of the matrix B_{r_1,r_2}. Using Lemma 4.17, this implies that $r_i \neq 0,\ \frac{m_i}{2},\ m_i,\ i = 1,2$. Conversely, suppose that $r_i \neq 0,\ \frac{m_i}{2},\ m_i,\ i = 1,2$. It is then not difficult to check that the system (4.18) is solvable.
(iii) Equivalently, we have to show that the system: $B'_{m_1,r_2}x + B'_{\frac{m_1}{2},r'_2}y = e$ is solvable under the conditions: $r_2 \neq 0, m_2$ and $r'_2 \neq \frac{m_2}{2}$. Indeed, this system reads

$$\begin{aligned} & x_2 + y_2 + y_3 = 1, \\ & (m_2 - r_2)(x_1 + x_2) + (m_2 - r'_2)(y_1 + y_2) + r'_2(y_3 + y_4) = 1, \\ & m_1 x_2 + \tfrac{m_1}{2}(y_1 + y_2 + y_3 + y_4) = 1, \\ & m_1(m_2 - r_2)(x_1 + x_2) + \tfrac{m_1 m_2}{2}(y_1 + y_2 + y_3 + y_4) = 1. \end{aligned}$$

One can easily check that it is solvable. (iv) can be checked in the same way. □

We can now characterize when the cut family $\mathcal{F}_{h,h'}$ is full dimensional, for $1 \leq h < h' \leq k$. To simplify the notation, we state the result for the indices $h = 1, h' = 2$.

PROPOSITION 4.19 *If $m_1 = m_2 = 1$, then the family $\mathcal{F}_{1,2}$ is full dimensional if and only if there exists a root pattern $r \in \mathcal{P}$ such that (r_1, r_2) is equal to $(1,0)$ or to $(0,1)$.*
If $m_1 = 1$ and $m_2 \geq 2$, then $\mathcal{F}_{1,2}$ is full dimensional if and only if there exists a root pattern $r \in \mathcal{P}$ such that $r_2 \neq 0, m_2$.
Suppose now that $m_1, m_2 \geq 2$. Then, $\mathcal{F}_{1,2}$ is full dimensional if and only if one of the following conditions (i)-(ii) *holds:*

(i) *There exists* $r \in \mathcal{P}$ *such that* $r_i \notin \{0, \frac{m_i}{2}, m_i\}$ *for* $i = 1, 2$.

(ii) *For every* $r \in \mathcal{P}$, $r_1 \in \{0, \frac{m_1}{2}, m_1\}$ *or* $r_2 \in \{0, \frac{m_2}{2}, m_2\}$ *and one of the conditions* (iia)-(iic) *holds:*

   (iia) $A_1 \neq \emptyset$ *and* $B_1 \neq \emptyset$.

   (iib) $A_2 \neq \emptyset$ *and* $B_2 \neq \emptyset$.

   (iic) $A_1 = A_2 = \emptyset$; *there exists* $r \in \mathcal{P}$ *such that* $r_1 = \frac{m_1}{2}$ *and* $r_2 \notin \{0, \frac{m_2}{2}, m_2\}$; *there exists* $r' \in \mathcal{P}$ *such that* $r'_1 \notin \{0, \frac{m_1}{2}, m_1\}$ *and* $r'_2 = \frac{m_2}{2}$.

The sets A_i, B_i *(*$i = 1, 2$*) are defined by*

$A_i := \{(r_1, r_2) \mid r \in \mathcal{P}$ *and* $r_i = 0, m_i\} \setminus \{(0,0), (0, m_2), (m_1, 0), (m_1, m_2)\}$,

$B_i := \{(r_1, r_2) \mid r \in \mathcal{P}$ *and* $r_i = \frac{m_i}{2}\} \setminus \{(\frac{m_1}{2}, \frac{m_2}{2})\}$.

Proof. If $m_1 = m_2 = 1$ then $\mathcal{F}_{1,2}$ is full dimensional if and only if it contains a nonzero vector, i.e., if there exists $r \in \mathcal{P}$ with $(r_1, r_2) = (0, 1)$ or $(1, 0)$. The case $m_1 = 1, m_2 \geq 2$ follows using Lemma 4.16 (i). We now suppose that $m_1, m_2 \geq 2$. If there exists $r \in \mathcal{P}$ such that $r_i \notin \{0, \frac{m_i}{2}, m_i\}$ for $i = 1, 2$, then $\mathcal{F}_{1,2}$ is full dimensional by Lemma 4.16 (ii). So, we now suppose that $r_1 \in \{0, \frac{m_1}{2}, m_1\}$ or $r_2 \in \{0, \frac{m_2}{2}, m_2\}$ for every $r \in \mathcal{P}$. Hence, for every $r \in \mathcal{P}$,

$$(r_1, r_2) \in A_1 \cup A_2 \cup B_1 \cup B_2 \cup \{(0,0), (0, m_2), (m_1, 0), (m_1, m_2), (\frac{m_1}{2}, \frac{m_2}{2})\}.$$

If $A_1 \neq \emptyset$ and $B_1 \neq \emptyset$, then $\mathcal{F}_{1,2}$ is full dimensional by Lemma 4.16 (iii). Similarly, $\mathcal{F}_{1,2}$ is full dimensional if $A_2 \neq \emptyset$ and $B_2 \neq \emptyset$. Therefore, we can now suppose that one of A_i and B_i is empty for $i = 1, 2$. We claim that

$$\mathcal{F}_{1,2} \text{ is full dimensional } \iff \text{(iic) holds.}$$

Suppose first that $\mathcal{F}_{1,2}$ is full dimensional. We show that $A_1 = A_2 = \emptyset$. By the above, we know that one of A_i or B_i is empty for $i = 1, 2$. If $A_1 = B_2 = \emptyset$, then using Lemma 4.17, we obtain that the set $\mathcal{F}_{1,2}$ is contained in the hyperplane H_1, in contradiction with its full dimensionality. Similarly, if $A_2 = B_1 = \emptyset$, the $\mathcal{F}_{1,2}$ is contained in the hyperplane H_2. Finally, if $B_1 = B_2 = \emptyset$, then $\mathcal{F}_{1,2}$ is contained into H_0. We obtain a contradiction with the full dimensionality of $\mathcal{F}_{1,2}$. This shows that $A_1 = A_2 = \emptyset$. If $\mathcal{P}$ contains no pattern r satisfying $r_1 = \frac{m_1}{2}$ and $r_2 \notin \{0, \frac{m_2}{2}, m_2\}$, then one deduces again from Lemma 4.17 that $\mathcal{F}_{1,2}$ is contained in the hyperplane H_2, yielding a contradiction. So, we have shown that (iic) holds under the assumption that $\mathcal{F}_{1,2}$ is full dimensional.
Conversely, let us suppose that (iic) holds. Let $r, r' \in \mathcal{P}$ as in (iic). We deduce from Lemma 4.16 (iv) that $\mathcal{F}_{1,2}$ is full dimensional. □

We conclude with a few remarks. We have seen in Lemma 4.1 that a necessary condition for the inequality (2.4) to be facet defining is that the matrix $M_{\mathcal{P}}$ have full column rank. Another similar necessary condition can be formulated in terms of the incidence matrix $A_{\mathcal{P}}$ of the set $\mathcal{P}$ of root patterns. More precisely, let $A_{\mathcal{P}}$ denote the matrix whose rows are the root patterns $r \in \mathcal{P}$. Hence, $A_{\mathcal{P}}$ has k columns and $|\mathcal{P}|$ rows.

PROPOSITION 4.20 *If the inequality* (2.4) *defines a facet of the cut polytope, then*

$$\text{rank } A_{\mathcal{P}} = k.$$

PROOF. Suppose that rank $A_{\mathcal{P}} \leq k-1$. We show that the inequality (2.4) is not facet defining. Consider the system of equations:

$$A_{\mathcal{P}} z = \frac{\sigma + \gamma}{2} e,$$

where e denotes the all ones vector. This system has at least one solution, namely, the vector $a := (a_1, \ldots, a_k)$ (recall (2.5)). As rank $A_{\mathcal{P}} \leq k-1$, we can find another solution $z \neq a$ of the system $A_{\mathcal{P}} x = \frac{\sigma+\gamma}{2} e$. Then, every root of (2.4) satisfies the equation:

$$\sum_{h \in K^*} (z_h)^2 \sum_{i<j, i,j \in V_h} x_{ij} + \sum_{1 \leq h < h' \leq k} z_h z_{h'} \sum_{i \in V_h, j \in V_{h'}} x_{ij} = \frac{\sigma^2 - \gamma^2}{4}.$$

This shows that (2.4) is not facet defining. □

We group below several necessary conditions that can be deduced from our results.

COROLLARY 4.21 *Suppose that the inequality* (2.4) *is facet defining. Then, the following conditions hold:*

(i) $|\mathcal{P}| \geq \binom{k}{2} + |K^*|$.

(ii) *Suppose $m_h = 1$ for some $h = 1, \ldots, k$. Then, there exist two root patterns r, r' such that $r_h = 0$ and $r'_h = 1$.*

(iii) *Suppose $m_h \geq 2$ for some $h = 1, \ldots, k$. Then, there exist two root patterns r, r' such that $r'_h \notin \{r_h, m_h - r_h\}$. Moreover, $m_h \geq a_1 \wedge \ldots \wedge a_{h-1} \wedge a_{h+1} \wedge \ldots \wedge a_k$.*

(iv) *Suppose that $m_h, m_{h'} \geq 2$ for some $h \neq h' \in \{1, \ldots, k\}$. Then, there exist two root patterns r, r' such that $r'_h - r'_{h'} \notin \{r_h - r_{h'}, m_h - r_h - m_{h'} + r_{h'}\}$.*

PROOF. (i) follows from Lemma 4.1. In what follows, we let e denote the all ones vector (of appropriate dimension).
(ii) Let $m_{h_0} = 1$. Suppose that there exists $\rho \in \mathbb{Z}$ such that $r_{h_0} = \rho$ for all $r \in \mathcal{P}$. Let $u \in \mathbb{R}^k$ be defined by $u_{h_0} = 1$ and $u_h = 0$ for $h \in \{1, \ldots, k\} \setminus \{h_0\}$. Then,

$$A_{\mathcal{P}} u = \rho e.$$

On the other hand,

$$A_{\mathcal{P}} a = \frac{\sigma + \gamma}{2} e.$$

This shows that rank $A_{\mathcal{P}} < k$. Therefore, (2.4) is not facet defining by Proposition 4.20.
(iii) Suppose that there exist $h_0 \in K^*$ and $\rho \in \mathbb{Z}$ such that $r_{h_0} \in \{\rho, m_{h_0} - \rho\}$ for all $r \in \mathcal{P}$. Let $v \in \mathbb{R}^J$ be defined by $v_{h_0 h_0} := 1$ and $v_{hh'} := 0$ for $hh' \in J \setminus \{h_0 h_0\}$. Then,

$$M_{\mathcal{P}} v = \rho(m_{h_0} - \rho) e.$$

On the other hand, defining $\alpha \in \mathbb{R}^J$ by setting $\alpha_{hh'} := a_h a_{h'}$ for $hh' \in J$, we have:

$$M_{\mathcal{P}} \alpha = \frac{\sigma^2 - \gamma^2}{4} e.$$

This shows that $M_{\mathcal{P}}$ does not have full column rank. Therefore, (2.4) is not facet defining, by Lemma 4.1. This shows the first part of (iii). Suppose now, for instance, that $2 \leq m_1 < a_2 \wedge \ldots \wedge a_k$. Let $r \neq r' \in \mathcal{P}$. Then, $0 = \sum_{1 \leq h \leq k} a_h (r_h - r'_h)$, which implies that $r_1 = r'_1$ as $a_2 \wedge \ldots \wedge a_k$ divides $r_1 - r'_1$ and $|r_1 - r'_1| \leq m_1$. We obtain a contradiction.
(iv) Let $1 \leq h_0 < h_1 \leq k$ such that $m_{h_0}, m_{h_1} \geq 2$. Suppose that there exists $\rho \in \mathbb{Z}$ such that $r_{h_1} - r_{h_0} \in \{\rho, m_{h_1} - m_{h_0} - \rho\}$ for all $r \in \mathcal{P}$. Let $w \in \mathbb{R}^J$ be defined by $w_{h_0 h_0} = w_{h_1 h_1} := -1$, $w_{h_0 h_1} := 1$ and $w_{hh'} := 0$ otherwise. Then,

$$M_{\mathcal{P}} w = \rho(\rho + m_{h_0} - m_{h_1}) e.$$

As in (iii), this shows that $M_{\mathcal{P}}$ does not have full column rank and, thus, that (2.4) is not facet defining. □

EXAMPLE 4.22 Consider the sequence $b := (1, 1, 1, 1, 1, 2, 2, 4, 4)$; its gap is equal to 1. This is the case: $n = 9$, $k = 3$, $a = (1, 2, 4)$ and $m = (5, 2, 2)$. There are four root patterns: $r = (3, 1, 1)$, $(1, 2, 1)$, $(5, 0, 1)$, and $(1, 0, 2)$. Hence, rank $A_{\mathcal{P}} = 3$. However, the inequality (2.4) is not facet defining in this case as there too few root patterns. Indeed, one needs at least $\binom{k}{2} + |K^*| = 6$ root patterns !

As another example, consider the sequence $b := (1, 1, 1, 1, 2, 2)$ with gap 0. There are only two root patterns: $(4, 0)$ and $(2, 1)$. Hence, (2.4) is not facet defining. □

5 Sequences with two and three values

In this section, we show that Conjecture 1.4 holds for sequences with two values. We start with a lemma which is a refinement of Lemma 3.5.

LEMMA 5.1 *Let $a_1, a_2 \geq 1$ be relatively prime integers. Then, there exist integers u_1, u_2 satisfying*

$$\begin{cases} u_1 a_1 - u_2 a_2 = 1, \\ 0 \leq u_1 \leq a_2, \ 0 \leq u_2 \leq a_1. \end{cases}$$

Moreover, if $a_1, a_2 \geq 2$, then u_1, u_2 can be chosen to satisfy

$$1 \leq u_1 \leq a_2 - 1, \ 1 \leq u_2 \leq a_1 - 1.$$

PROOF. As $a_1 \wedge a_2 = 1$, we can find integers $u_1, u_2 \geq 0$ such that $u_1 a_1 - u_2 a_2 = 1$. Choose such u_1, u_2 in such a way that $\max(u_1, u_2)$ is minimum. Then, $u_1 \leq a_2$. Indeed, suppose $u_1 > a_2$. This implies $u_2 > a_1$. Indeed, $u_1 a_1 = 1 + a_2 u_2 > a_2 a_1$, which yields $u_2 \geq a_1$. But, then, $u'_1 := u_1 - a_2$, $u'_2 := v_2 - a_1$ are nonnegative integers satisfying $u'_1 a_1 - u'_2 a_2 = 1$, which contradicts the minimality of $\max(u_1, u_2)$. Therefore, $u_1 \leq a_2$; this implies $u_2 < a_1$ as $u_2 a_2 = u_1 a_1 - 1 < a_1 a_2$. Moreover, $u_1 = 0$, or $u_1 = a_2$, or $u_2 = 0$ can occur only if one of a_1, a_2 is equal to 1. □

THEOREM 5.2 *Let $a_1, a_2 \geq 1$ be relatively prime integers. Let $b \in \mathbb{Z}^n$ take the two values a_1 and a_2 with respective multiplicities m_1 and m_2. If the inequality* (2.4) *defines a facet of* CUT_n*, then $\gamma(b) = 1$.*

PROOF. We first rule out the case when one of m_1 or m_2 is equal to 1. Say, $m_1 = 1$. If (2.4) is facet defining then, by Corollary 4.21, there are at least two root patterns $r = (0, r_2)$ and $r' = (1, r'_2)$. Hence, $0 = a_1(r'_1 - r_1) + a_2(r'_2 - r_2)$, i.e., $0 = a_1 + a_2(r'_2 - r_2)$. This implies that a_2 divides a_1, i.e., that $a_2 = 1$. Then, $r'_2 = r_2 - a_1$ which yields $a_1 \leq m_2$. Therefore, the sequence $b = (a_1, \underbrace{1, \ldots, 1}_{m_2})$ has gap $\gamma(b) \leq 1$. From now on, we can suppose that $m_1, m_2 \geq 2$.

Let v_1, v_2 be integers satisfying $v_1 a_1 - v_2 a_2 = 1$ and $0 \leq v_1 \leq a_2$, $0 \leq v_2 \leq a_1$. Set $w_1 := a_2 - v_1$ and $w_2 := a_1 - v_2$. Then, $-a_1 w_1 + a_2 w_2 = 1$. Suppose that b has gap $\gamma \geq 2$. Then, applying Lemma 3.3, we obtain:
(a) $r_1 < v_1$ or $m_2 - r_2 < v_2$, and
(b) $m_1 - r_1 < w_1$ or $r_2 < w_2$.
There are four cases to be considered.

Case 1: $r_1 < v_1$ and $m_1 - r_1 < w_1$.
Then, $m_1 < v_1 + w_1 = a_2$. By Corollary 4.21, we deduce that (2.4) is not facet inducing.

Case 2: $m_2 - r_2 < v_2$ and $r_2 < w_2$.
Then, $m_2 < a_1$, which implies as before that (2.4) is not facet defining.

Case 3: $r_1 < v_1$ and $r_2 < w_2$.
Suppose that there is another pattern $r' = (r'_1, r'_2)$ for the roots. Then, from $a_1(r_1 - r'_1) + a_2(r_2 - r'_2) = 0$, we obtain

$$r'_1 = r_1 + pa_2,$$
$$r'_2 = r_2 - pa_1.$$

for some $p \in \mathbb{Z}$. Hence, $p = \frac{r'_1 - r_1}{a_2} \geq -\frac{r_1}{a_2} > -\frac{v_1}{a_2} \geq -1$ and $p = \frac{r_2 - r'_2}{a_1} \leq \frac{r_2}{a_1} < \frac{w_2}{a_1} \leq 1$. Therefore, $p = 0$. This shows that $r' = r$, i.e., there is a unique pattern for the roots of (2.4). Therefore, (2.4) is not facet inducing.

Case 4: $m_2 - r_2 < v_2$ and $m_1 - r_1 < w_1$.
This case is similar to Case 3. □

We can also show that Conjecture 1.4 holds for some sequences with three values. We state a preliminary result.

LEMMA 5.3 *Suppose that a_1, a_2, a_3 are pairwise relatively prime. If the gap satisfies $\gamma \geq 2$, then $r_i \neq r'_i$ for $i = 1, 2, 3$, for any two distinct root patterns.*

PROOF. Let r, r' be two distinct root patterns such that $r_3 = r'_3$. From the relation $\sum_{1 \leq i \leq 3} a_i(r_i - r'_i) = 0$ follows that $a_1(r_1 - r'_1) + a_2(r_2 - r'_2) = 0$. As $a_1 \wedge a_2 = 1$, we deduce that

$$r_1 = r'_1 + za_2, \ r_2 = r'_2 - za_1$$

for some integer z. We can suppose, for instance, that $z \geq 1$. Then, $1 \leq z = \frac{r_1 - r'_1}{a_2} \leq \frac{r_1}{a_2}$ and $1 \leq z = \frac{r'_2 - r_2}{a_1} \leq \frac{m_2 - r_2}{a_1}$, i.e., $r_1 \geq a_2$ and $m_2 - r_2 \geq a_1$. Let u_1, u_2 be integers such that $u_1 a_1 - u_2 a_2 = 1$, $0 \leq u_1 \leq a_2$ and $0 \leq u_2 \leq a_1$. Then, the assumption of Lemma 3.3 holds as $r_1 \geq a_2 \geq u_1$ and $m_2 - r_2 \geq a_1 \geq u_2$. Therefore, the gap is 0 or 1 by Lemma 3.3. □

PROPOSITION 5.4 *Conjecture 1.4 holds for any sequence taking the values a_1, a_2 and $a_3 := a_1 + 1$, where $a_1, a_2 \geq 1$ are integers such that $a_1 \wedge a_2 = (a_1 + 1) \wedge a_2 = 1$.*

PROOF. Supose that the gap satisfies $\gamma \geq 2$. Note that $ua_1 + v(a_1 + 1) = 1$ holds for $(u, v) = (-1, 1)$. Applying Lemma 3.3, we obtain that every root pattern r satisfies: $m_1 - r_1 < 1$ or $r_3 < 1$, i.e., $r_1 = m_1$ or $r_3 = 0$. Using Lemma 5.3, this implies that there are at most two distinct root patterns. Therefore, the inequality (1.2) is not facet defining by Corollary 4.21. □

We conclude with some examples.

EXAMPLE 5.5 *Let* $a_1 := 2,\ a_2 := 3,\ a_3 := 7$. *Conjecture* 1.4 *holds for this sequence, i.e., the gap is 0 or 1, or the inequality* (2.4) *does not define a facet of the cut polytope.*

We distinguish 7 cases, according to the respective parities of m_1, m_2, m_3. (We indicate in each case what is the suitable partition realizing the minimum gap.)
• m_1, m_2 even, m_3 odd; then $\gamma = 1$, as the sequence $(2,2,3,3,7)$ has gap 1 (with partition: 27; by this we mean the partition with 2,7 on one side and 2,3,3 on the other side).
• m_1, m_3 even, m_3 odd; then, $\gamma = 1$ as the sequence $(2,2,3,7,7)$ has gap 1 (with partition: 227).
• The sequence $(2,3,3,7,7)$ has gap 2, but does not define a facet of CUT_5. This is the smallest case of m_1 odd, m_2, m_3 even. The next cases to consider are:
- $m_1 = 3,\ m_2 = m_3 = 2$; then, the sequence $(2,2,2,3,3,7,7)$ has gap 0 (with partition: 2227).
- $m_1 = 1,\ m_2 = 4,\ m_3 = 2$; then, the sequence $(2,3,3,3,3,7,7)$ has gap 0 (with partition: 77).
- $m_1 = 1, m_2 = 2,\ m_3 = 4$; then, the sequence $(2,3,3,7,7,7,7)$ has gap 2. In fact, the sequence $(2,3,3,\underbrace{7,\ldots,7}_{2m})$ has gap 2. The roots are the subsets S satisfying $b(S) = \frac{\sigma+2}{2} = 5 + 7m$. Hence, there is only one root pattern, namely, $r = (1,1,m)$. Therefore, the corresponding inequality (2.4) does not define a facet of the cut polytope.
• m_1 even, m_2, m_3 odd; then, $\gamma = 0$ as the sequence $(2,2,3,7)$ has gap 0 (with partition: 7).
• m_2 even, m_1, m_3 odd; then, $\gamma = 1$ as the sequence $(2,3,3,7)$ has gap 1 (with partition: 233).
• m_3 even, m_1, m_2 odd; then, $\gamma = 1$ as the sequence $(2,3,7,7)$ has gap 1 (with partition: 37).
• In the case m_1, m_2, m_3 odd, the gap is $\gamma = 0$ except $\gamma = 2$ for the sequences $(2,3,\underbrace{7,\ldots,7}_{2m+1})$ (then, $(0,0,m+1)$ is the only root pattern) and $(2,2,2,3,\underbrace{7,\ldots,7}_{2m+1})$ (then, $(3,1,m)$ and $(1,0,m+1)$ are the only root patterns). □

EXAMPLE 5.6 *Let* $a_1 := 2,\ a_2 := 3,\ a_3 := 5$. *Conjecture* 1.4 *holds for this sequence, i.e., the gap is 0 or 1, or the inequality* (2.4) *does not define a facet of the cut polytope.* We proceed as in Example 5.5.
• The sequence $(2,2,3,3,5)$ has gap 1 (with partition: 35).
• The sequence $(2,2,3,5,5)$ has gap 1 (with partition: 225).
• The sequence $(2,2,2,3,3,5,5)$ has gap 0 (with partition: 335) as well as the sequence $(2,3,3,3,3,5,5,)$ (with partition: 3333). On the other hand, the sequence

$(2,3,3,\underbrace{5,\ldots,5}_{2m})$ has gap 2. The roots are the sets S satisfying $b(S) = \frac{\sigma+2}{2} = 5(m+1)$. Hence, there are two possible root patterns, namely, $r = (0,0,m+1)$ and $(1,1,m)$. Hence, we are in the nonfacet case by Corollary 4.21.

• The sequence $(2,3,3,5)$ has gap 1 (with partition: 25).

• The sequence $(2,3,5,5)$ has gap 1 (with partition: 35).

• The sequence $(2,2,2,2,3,5)$ has gap 0 (with partition: 2222) as well as the sequence $(2,2,3,3,3,5)$ (with partition: 333). On the other hand, the sequence $(2,2,3,\underbrace{5,\ldots,5}_{2m+1})$ has gap 2. The roots should satisfy $b(S) = 7+5m$; hence, the possible root patterns are $r = (1,0,m+1)$ and $(2,1,m)$. Hence, the inequality (2.4) is not facet defining in this case.

• The sequence $(2,3,5)$ has gap 0 (with partition: 23). $\square$

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