TRUST REGION PROBLEMS AND NONSYMMETRIC PERTURBATION THEORY

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Abstract

A characterization is given for the spectrum of a symmetric matrix to remain real after a nonsymmetric sign restricted border perturbation, including the case where the perturbation is skew-symmetric. The characterization is in terms of the stationary points of a quadratic function on the unit sphere. This yields interlacing relationships between the eigenvalues of the original matrix and those of the perturbed matrix. As a result of the linkage between the perturbation and stationarity problems, new theoretical insights are gained for each. Applications of the main results include a characterization of those matrices which are exponentially nonnegative with respect to the n-dimensional ice-cream cone, which in turn leads to a decomposition theorem for such matrices. In addition, results are obtained for nonsymmetric matrices regarding interlacing and majorization.

Key words: trust region problems, nonsymmetric perturbation, secular function, secular antiderivative, eigenvalues, interlacing, exponential nonnegativity, majorization, inverse eigenvalue problems.

1 Introduction

Suppose that B is a real symmetric $(n-1) \times (n-1)$ matrix. Then the classical Raleigh Principle and Courant-Fischer Minimax Theorem relate the eigenvalues of B to the stationary points of the quadratic function

$$\nu(x) = x^t Bx$$

with respect to the constraint set

$$S_{n-1} = \{x \in R^{n-1} : x^t x = 1\}.$$

In particular, if we introduce the *Lagrangian function*

$$L(x, \lambda) = \nu(x) - \lambda x^t x, \tag{1.1}$$

then the *Lagrange equation*

$$\partial_x L(x, \lambda) = 0 \tag{1.2}$$

becomes

$$Bx - \lambda x = 0. \tag{1.3}$$

If x and λ satisfy the Lagrange equation and $x \in S_{n-1}$, then we shall say that λ and x are a *Lagrange multiplier* and an associated *stationary point* of $\nu(\cdot)$ with respect to S_{n-1}, respectively. Thus there is a one-to-one correspondence between the eigenvalues of B and the Lagrange multipliers. Furthermore, the stationary points, including the maximum and minimum points, can be found by determining the unit eigenvectors of B. Lack of convexity does not cause difficulties in locating the constrained maxima and minima, since such points correspond to the maximum and minimum eigenvalues, respectively.

The eigenvalues of symmetric border perturbations of B have well-known properties. In particular, the eigenvalues $\delta_1 \geq \delta_2 \ldots \geq \delta_n$ of the $n \times n$ matrix

$$A = \begin{pmatrix} B & \eta \\ \eta^t & t \end{pmatrix} \tag{1.4}$$

interlace the eigenvalues of B, which we denote $\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_{n-1}$. That is,

$$\delta_1 \geq \gamma_1 \geq \delta_2 \geq \gamma_2 \geq \ldots \geq \gamma_{n-1} \geq \delta_n. \tag{1.5}$$

(See e.g. pages 94–97 in Wilkinson [23].)

Nonsymmetric border perturbations of B are not as well understood. For example, the $n \times n$ matrix

$$A = \begin{pmatrix} B & -\alpha \\ \alpha^t & t \end{pmatrix}, \tag{1.6}$$

which is a skew-symmetric perturbation for $t = 0$, may possess either a complex or real spectrum, and may be either diagonalizable or derogatory.

On the other hand, the important problem of finding the Lagrange multipliers and stationary points of the general quadratic function

$$\mu(x) = x^t B x - 2\eta^t x \tag{1.7}$$

on S_{n-1} has been extensively studied in the literature. In particular, we shall consider the "trust region" problems

$$P^{\mu}_{\min}: \quad \min\{\mu(x) \ : \ x \in S_{n-1}\}$$

and

$$P^{\mu}_{\max}: \quad \max\{\mu(x) \ : \ x \in S_{n-1}\}.$$

Such problems arise during the calculation of the step between iterates in an important class of minimization algorithms called "trust region methods". (The step in trust region algorithms is actually calculated with a constraint of the form $\| Gy \| \leq \psi$, for some nonsingular matrix G and $\psi > 0$. However, complementary slackness and the change of variables $x = (1/\psi)Gy$ lead to the form of our trust region problems.) The theory has been discussed in Forsythe and Golub [5], Golub [9], Gander [6], Sorensen [20], Fletcher [4] and Gander, Golub and Von Matt [7]. Furthermore, numerical techniques for solving trust region problems are given in [20], Moré and Sorensen [17], [4], Coleman and Hempel [3], and [7].

In the present work, we connect spectral properties of a nonsymmetrically perturbed symmetric matrix to stationarity properties of a specific trust region problem. We provide explicit criteria for the spectrum of the perturbed matrix to remain real, as well as eigenvalue interlacing properties. We shall consider certain sign restricted nonsymmetric border perturbations, including the case (1.6). Our approach, in essence, is to regard the perturbation of a matrix as a linear perturbation of a purely quadratic form. As a result of the interplay between the trust region and perturbation problems, new theoretical insights are gained for each.

In the next section we summarize required known facts concerning trust region problems, some of which involve the so-called *secular function* associated with $\mu(\cdot)$. In addition, we shall make use of the *secular antiderivative function* associated with $\mu(\cdot)$. This is a key tool which we employ in order to relate results on trust region problems to perturbation theory. The main results are then given in Section 3, including interlacing relationships for a nonsymmetrically perturbed matrix.

Section 4 contains applications of our main results. These include a characterization of matrices which are exponentially nonnegative with respect to the n-dimensional ice-cream cone, which leads to a decomposition theorem for such matrices. In addition, results are givend for nonsymmetric matrices regarding interlacing and majorization.

2 Trust Region Problems

2.1 Some Known Results

For the real symmetric $(n-1)\times(n-1)$ matrix B and the real $(n-1)$-vector η, consider the quadratic function $\mu(\cdot)$ given by (1.7) on S_{n-1}. Then the Lagrangian function is

$$L(x,\lambda) = x^tBx - 2\eta^tx - \lambda x^tx. \tag{2.1}$$

In all that follows, our terminology regarding Lagrange multipliers and stationary points is as in Section 1, with the appropriate Lagrange equation replacing (1.3). Presently, the Lagrange equation is

$$(B-\lambda I)x - \eta = 0, \tag{2.2}$$

The set of Lagrange multipliers of $\mu(\cdot)$ with respect to S_{n-1} will be denoted by Λ, and for $\lambda \in \Lambda$, the associated set of stationary points will be denoted by $S_\mu(\lambda)$.

Useful properties concerning trust region problems are summarized in the following theorem.

Theorem 2.1

1. *The vector $x \in R^{n-1}$, with $x^tx = 1$, is a minimum (maximum) point of $\mu(\cdot)$ over S_{n-1} if and only if there exists a scalar λ such that x and λ together satisfy the Lagrange equation (2.2), with the matrix $B-\lambda I$ being positive (negative) semidefinite.*

2. *The set Λ of Lagrange multipliers of $\mu(\cdot)$ with respect to S_{n-1} is finite. Let Λ be given by*

$$\lambda_1 > \lambda_2 > \ldots > \lambda_k,$$

and let $x^{\lambda_i} \in S_\mu(\lambda_i), \quad i = 1, 2, \ldots, k$. Then

$$\mu(x^{\lambda_1}) > \mu(x^{\lambda_2}) > \ldots > \mu(x^{\lambda_k}).$$

In particular, the minimum (maximum) of $\mu(\cdot)$ over S_{n-1} is attained at any stationary point associated with λ_k (λ_1).

Part 1 of the above theorem is due to Sorensen [20]; see also pages 101–102 in Fletcher [4]. Part 2 is due to Forsythe and Golub [5]; see also the discussion of Case(b) below.

Again denoting the spectrum of B by $\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_{n-1}$, let P be an orthogonal matrix such that

$$P^t BP = D = \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_{\mathrm{n}-1}), \tag{2.3}$$

the diagonal matrix with diagonal elements $\gamma_1, \gamma_2, \ldots, \gamma_{n-1}$. Then the Lagrange equation (2.2) becomes

$$(D - \lambda I)\hat{x} = \hat{\eta}, \tag{2.4}$$

where

$$\hat{x} = P^t x, \quad \hat{\eta} = P^t \eta.$$

The set of Lagrange multipliers Λ is not changed by this transformation of the Lagrange equation, and for every $x \in R^{n-1}$ we have

$$\hat{\mu}(\hat{x}) := \hat{x}^t D\hat{x} - 2\hat{\eta}^t \hat{x} = \mu(x). \tag{2.5}$$

We now introduce the condition

$$\hat{\eta}_i \neq 0 \ \ \forall i = 1, 2, \ldots, n-1, \tag{2.6}$$

or equivalently,

$$\textit{no eigenvector of } B \textit{ is orthogonal to } \eta. \tag{2.7}$$

There are two cases to consider.

Case a. *Condition (2.6) holds.* Then (2.4) implies that if $\lambda \in \Lambda$, it must be the case that $D - \lambda I$ and $B - \lambda I$ are invertible. Also, the sets $S_\mu(\lambda)$ and $S_{\hat{\mu}}(\lambda)$ are then the singletons

$$x^\lambda = (B - \lambda I)^{-1}\eta \tag{2.8}$$

and

$$\hat{x}^\lambda = (D - \lambda I)^{-1}\hat{\eta} = P^t x^\lambda, \tag{2.9}$$

respectively. Furthermore, in the present case, Λ is the set of solutions to the *implicit secular equation*

$$1 - \eta^t (B - \lambda I)^{-2}\eta = 0, \tag{2.10}$$

which has the same solution set as the *explicit secular equation*

$$f_\mu(\lambda) := 1 - \sum_{i=1}^{n-1} \left(\frac{\hat{\eta}_i}{\gamma_i - \lambda} \right)^2 = 0. \tag{2.11}$$

Continuing to utilize the terminology of [7], we shall call $f_\mu(\cdot)$ the *secular function* associated with $\mu(\cdot)$.

In Case a, unique solutions to P^μ_{min} and P^μ_{max} are given by x^{λ_k} and x^{λ_1}, respectively, from formula (2.8). Furthermore, in view of Part 1 of Theorem 2.1, the invertibility of $B - \lambda_k I$ implies

$$\lambda_k < \gamma_{n-1}, \tag{2.12}$$

while the invertibility of $B - \lambda_1 I$ implies

$$\lambda_1 > \gamma_1. \tag{2.13}$$

Case b. *Condition (2.6) does not hold.* In [5] it was proven that in this case, $\Lambda = \tilde{\Lambda} \cup \Gamma$, with $\tilde{\Lambda}$ being the solution set of the explicit secular equation, and where

$$\Gamma = \{\gamma_i \ : \ f_\mu(\gamma_i) > 0, \ 1 \leq i \leq n-1\},$$

where we adopt the convention $0/0 = 0$ in defining $f_\mu(\cdot)$. For each $\lambda \in \tilde{\Lambda}$, $B - \lambda I$ is invertible and $S_\mu(\lambda)$ is the singleton x^λ given by formula (2.8). For each $\gamma_i \in \Gamma$, the set $S_\mu(\gamma_i)$ is an $(m_{\gamma_i} - 1)$-dimensional manifold, where m_{γ_i} is the multiplicity of the eigenvalue γ_i.

In Case b, it is possible that $\gamma_1 = \lambda_1$, implying that λ_1 occurs strictly to the right of the maximal root of $f_\mu(\cdot)$. Note that this can happen only if

$$\gamma_i = \lambda_1 \Longrightarrow \hat{\eta}_i = 0.$$

Furthermore, it is then possible that $f_\mu(\lambda_1) > 0$, implying that $S_\mu(\lambda_1)$, the set of solutions to the trust region problem $P^\mu_{\max}$, is not a singleton. Likewise, it is possible that $\gamma_{n-1} = \lambda_k$, implying that λ_k occurs strictly to the left of the minimal root of $f_\mu(\cdot)$. This is possible only if

$$\gamma_i = \lambda_k \Longrightarrow \hat{\eta}_i = 0.$$

It may then happen that $f_\mu(\lambda_k) > 0$, implying that $S_\mu(\lambda_k)$, the set of solutions to the trust region problem $P^\mu_{\min}$, is not a singleton.

2.2 The Secular Antiderivative

For the general quadratic function $\mu(\cdot)$ given by (1.7), consider the function

$$g_\mu(\lambda) = \lambda - \sum_{i=1}^{n-1} \left(\frac{\hat{\eta}_i^2}{\gamma_i - \lambda} \right), \tag{2.14}$$

with the convention $0/0 = 0$. Then the singularities of $g_\mu(\cdot)$ are the same as those of the secular function $f_\mu(\cdot)$, and what is more,

$$g'_\mu(\lambda) = f_\mu(\lambda) \tag{2.15}$$

at every non-singularity λ. We shall call $g_\mu(\cdot)$ the *secular antiderivative function* associated with $\mu(\cdot)$.

The following lemma will be used to establish connections between trust region problems and perturbation theory. The lemma asserts that in Case a, the secular antiderivative's values on the Lagrange multiplier set Λ are precisely the values of $\mu(\cdot)$ on the corresponding set of stationary points, as given by (2.8). A variant of this result may be found in Section 2 of Forsythe and Golub [5], where it is used in proving Part 2 of Theorem 2.1 above.

Lemma 2.1 *Assume that condition (2.7) holds (i.e. Case a), and let $\lambda \in \Lambda$. Then*

$$g_\mu(\lambda) = \mu(x^\lambda), \tag{2.16}$$

where $x^\lambda = (B - \lambda I)^{-1}\eta$.

Proof. Using the fact that $(\hat{x}^\lambda)^t \hat{x}^\lambda = 1$, we obtain

$$\begin{aligned} g_\mu(\lambda) &= \sum_{i=1}^{n-1} \left[\lambda \left(\frac{\hat{\eta}_i}{\gamma_i - \lambda} \right)^2 - \frac{\hat{\eta}_i^2}{(\gamma_i - \lambda)} \right] \\ &= \sum_{i=1}^{n-1} (2\lambda - \gamma_i) \left(\frac{\hat{\eta}_i}{\gamma_i - \lambda} \right)^2 \\ &= \sum_{i=1}^{n-1} \gamma_i \left(\frac{\hat{\eta}_i}{\gamma_i - \lambda} \right)^2 - 2 \sum_{i=1}^{n-1} \left(\frac{\hat{\eta}_i^2}{\gamma_i - \lambda} \right) \\ &= \hat{\mu}(\hat{x}^\lambda) = \mu(x^\lambda). \end{aligned}$$

□

At this point it will be useful to discuss the graph of the function $g_\mu(\cdot)$ in Case a. Clearly $g_\mu(\cdot)$ possesses a singularity at each eigenvalue γ_i, $i = 1, 2, \ldots, n-1$. Also, $g_\mu(\lambda) \to \infty$ as $\lambda \downarrow \gamma_i$, while $g_\mu(\lambda) \to -\infty$ as $\lambda \uparrow \gamma_i$, for each $i = 1, 2, \ldots, n-1$. Let i be such that $\gamma_{i+1} < \gamma_i$. Then there is at least one root of $g_\mu(\cdot)$ in (γ_{i+1}, γ_i). It is readily checked that $g_\mu'''(\lambda) < 0$, and consequently $g_\mu''(\lambda)$ is monotone decreasing in this interval. It follows that $g_\mu(\cdot)$ has at most one point of inflection on (γ_{i+1}, γ_i), which is possibly also a critical point. Should there be a point of inflection in (γ_{i+1}, γ_i), then on that interval $g_\mu(\cdot)$ is strictly convex to the left of this point, and strictly concave to the right of it. Hence $g_\mu(\cdot)$ has either zero, one, or two critical points on (γ_{i+1}, γ_i), with the possibility of only one critical point being accounted for by the existence of an inflection which is also critical. By again considering $g_\mu''(\cdot)$, we find that $g_\mu(\cdot)$ is strictly convex on the semi-infinite interval (γ_1, ∞), while we have strict concavity on the other semi-infinite interval, namely $(-\infty, \gamma_{n-1})$. Now, since $g_\mu(\lambda) \to \infty$ as $\lambda \downarrow \gamma_1$ and as $\lambda \to \infty$, we conclude that $g_\mu(\cdot)$ has a unique critical point, namely λ_1, on (γ_1, ∞). Similarly, since $g_\mu(\lambda) \to -\infty$ as $\lambda \uparrow \gamma_{n-1}$ and as $\lambda \to -\infty$, we see that $g_\mu(\cdot)$ has a unique critical point, namely λ_k, on $(-\infty, \gamma_{n-1})$. (Note that this agrees with (2.12) and (2.13).)

In Case a, it is clear that the set of critical points of the secular antiderivative function $g_\mu(\cdot)$ is Λ. Furthermore, in view of our previous discussion, we then have

$$\mu_1 = g_\mu(\lambda_1) > \mu_2 = g_\mu(\lambda_2) > \ldots > \mu_k = g_\mu(\lambda_k), \tag{2.17}$$

where we have adopted the notation

$$\mu_i = \mu(x^{\lambda_i}), \; i = 1, 2, \ldots, k$$

for the *stationary values* of $\mu(\cdot)$ on S_{n-1}.

The preceeding discussion is summarized in Fig. 1, which illustrates the graph of a typical secular antiderivative function when (2.7) holds.

Figure 1: The secular antiderivative

3 Main Results

In what follows, we will be considering the border perturbation of the real symmetric $(n-1) \times (n-1)$ matrix B given by

$$A = \begin{pmatrix} B & -\alpha \\ \beta^t & t \end{pmatrix}, \tag{3.1}$$

where α and β are real $(n-1)$-vectors and $t \in R$. Letting P be an orthogonal matrix which diagonalizes B as in (2.3), we define

$$\hat{P} = \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.2}$$

Then

$$\hat{A} := \hat{P}^t A \hat{P} = \begin{pmatrix} D & -\hat{\alpha} \\ \hat{\beta}^t & t \end{pmatrix}, \tag{3.3}$$

where

$$\hat{\alpha} = P^t \alpha, \quad \hat{\beta} = P^t \beta.$$

Let us assume that

$$\hat{\alpha}_i \hat{\beta}_i \geq 0 \ \ \forall i = 1, 2, \ldots, n-1. \tag{3.4}$$

Since a permutation can be built into P, we can without loss of generality assume that

$$\hat{A} = \begin{pmatrix} \tilde{D} & 0 & -\tilde{\alpha} \\ 0 & \bar{D} & -\bar{\alpha} \\ \tilde{\beta}^t & \bar{\beta}^t & t \end{pmatrix}, \tag{3.5}$$

where

$$\begin{aligned} \tilde{D} &= \operatorname{diag}(\tilde{\gamma}_1, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_{\tilde{n}}), \\ \bar{D} &= \operatorname{diag}(\bar{\gamma}_1, \bar{\gamma}_2, \ldots, \bar{\gamma}_{\bar{n}}), \\ \tilde{\alpha}_i \tilde{\beta}_i &= 0 \ \ \forall i = 1, 2, \ldots, \tilde{n}, \\ \bar{\alpha}_i \bar{\beta}_i &> 0 \ \ \forall i = 1, 2, \ldots, \bar{n}, \end{aligned}$$

and

$$\tilde{n} + \bar{n} = n - 1.$$

Furthermore, we can assume the ordering

$$\bar{\gamma}_1 \geq \bar{\gamma}_2 \geq \cdots \geq \bar{\gamma}_{\bar{n}}.$$

Remark 3.1

1. Note that condition (3.4) holds if $\alpha = \beta$; that is, when A is given by (1.6).

2. It is possible, of course, that either $\tilde{n}$ or $\bar{n}$ may be zero. It is readily shown that

$$\tilde{n} = 0 \Longleftrightarrow \textit{no eigenvector of } B \textit{ is orthogonal to } \alpha \textit{ or } \beta. \tag{3.6}$$

Consider the submatrix of $\hat{A}$ given by

$$\bar{A} = \begin{pmatrix} \bar{D} & -\bar{\alpha} \\ \bar{\beta}^t & t \end{pmatrix}. \tag{3.7}$$

We associate with $\bar{A}$ the quadratic function

$$\bar{\mu}(\bar{x}) = \bar{x}^t \bar{D} \bar{x} - \sum_{i=1}^{\bar{n}} (\bar{\alpha}_i \bar{\beta}_i)^{1/2} \bar{x}_i. \tag{3.8}$$

The secular antiderivative function associated with $\bar{\mu}(\cdot)$ is then

$$g_{\bar{\mu}}(\lambda) = \lambda - \sum_{i=1}^{\bar{n}} \left(\frac{\bar{\alpha}_i \bar{\beta}_i}{\bar{\gamma}_i - \lambda} \right). \tag{3.9}$$

From the structure of $\hat{A}$, we see that the characteristic polynomial of A is

$$p(\lambda) = \bar{p}(\lambda) \prod_{i=1}^{\tilde{n}} (\tilde{\gamma}_i - \lambda), \tag{3.10}$$

where

$$\bar{p}(\lambda) = \det(\bar{A} - \lambda \bar{I}). \tag{3.11}$$

In view of (3.10), it is clear that each of the $\tilde{n}$ diagonal entries of $\tilde{D}$ is an eigenvalue of A. Therefore in order to completely determine the spectrum of A, it is necessary only to determine the spectrum of $\bar{A}$. The following key lemma describes this spectrum in terms of the secular antiderivative function associated with $\bar{\mu}(\cdot)$, and is the basis of our linkage between trust region problems and perturbation theory.

Lemma 3.1 *The distinct real eigenvalues of $\bar{A}$ which differ from the $\bar{n}$ values $\bar{\gamma}_i$ are the solutions of*

$$g_{\bar{\mu}}(\lambda) = t. \tag{3.12}$$

Proof. Let $\lambda \in R$ where $\lambda \neq \bar{\gamma}_i \quad \forall i = 1, 2, \ldots, \bar{n}$. From the Schur complement formula we then obtain

$$\det(\bar{A} - \lambda I) = \det(\bar{D} - \lambda I)[t - \lambda + \bar{\beta}^t(\bar{D} - \lambda I)^{-1}\bar{\alpha}], \tag{3.13}$$

from which it follows that the distinct real eigenvalues of $\bar{A}$ differing from the $\bar{n}$ numbers $\bar{\gamma}_i$ are the solutions of

$$t - \lambda + \sum_{i=1}^{\bar{n}} \left(\frac{\bar{\alpha}_i \bar{\beta}_i}{\bar{\gamma}_i - \lambda} \right) = 0. \tag{3.14}$$

In view of (3.9), this is equivalent to (3.12).

□

Let us denote the set of Lagrange multipliers of $\bar{\mu}(\cdot)$ with respect to $S_{\bar{n}}$ by $\bar{\Lambda}$, and let this set be given by

$$\bar{\lambda}_1 > \bar{\lambda}_2 > \ldots > \bar{\lambda}_m.$$

Since condition (2.6) holds for $\bar{\mu}(\cdot)$, we are presently in Case a. Therefore the set of critical points of $g_{\bar{\mu}}(\cdot)$ is $\bar{\Lambda}$, and moreover, in view of (2.17) we have

$$\bar{\mu}_1 = g_{\bar{\mu}}(\bar{\lambda}_1) > \bar{\mu}_2 = g_{\bar{\mu}}(\bar{\lambda}_2) > \ldots > \bar{\mu}_m = g_{\bar{\mu}}(\bar{\lambda}_m), \tag{3.15}$$

where the stationary values of $\bar{\mu}(\cdot)$ on $S_{\bar{n}}$ are denoted

$$\bar{\mu}_i = \bar{\mu}(\bar{x}^{\bar{\lambda}_i}), \ i = 1, 2, \ldots, m$$

Here

$$\bar{x}^{\bar{\lambda}_i} = (\bar{D} - \bar{\lambda}_i \bar{I})^{-1}\bar{\eta}, \ i = 1, 2, \ldots, m,$$

with $\bar{\eta}$ being the $\bar{n}$-vector whose i^{th} component is $\bar{\alpha}_i \bar{\beta}_i$.

The next theorem provides a qualitative description of the eigenstructure of the matrix A given by (3.1), when condition (3.4) holds. Realness of the spectrum of A is characterized in terms of the graph of $g_{\bar{\mu}}(\cdot)$, and in particular, in terms of the stationary values of the quadratic function $\bar{\mu}(\cdot)$. Should the spectrum be real, the interlacing relationships between the eigenvalues of A and B are described. Prior to stating the result, we require some further terminology and notation.

Let $\bar{\lambda} \in \bar{\Lambda}$. We shall say that $\bar{\lambda}$ is a *type-1 critical point* of $g_{\bar{\mu}}(\cdot)$ if it is a critical point which is also an inflection. Otherwise we call $\bar{\lambda}$ a *type-2 critical point* of $g_{\bar{\mu}}(\cdot)$. From the discussion of the secular antiderivative function given in Section 2.2, it is clear that the number of type-2 critical points is even, since these points occur pairwise upon the particular bounded intervals $(\bar{\gamma}_{i+1}, \bar{\gamma}_i)$ where they exist, and in addition, there is a single type-2 critical point in each of the semi-infinite intervals $(-\infty, \bar{\gamma}_{\bar{n}})$ and $(\bar{\gamma}_1, \infty)$; these are $\bar{\lambda}_{\bar{n}}$ and $\bar{\lambda}_1$, respectively. Let us denote the sets of type-1 and type-2 critical points of $g_{\bar{\mu}}(\cdot)$ by $\bar{\Lambda}'$ and $\bar{\Lambda}''$, respectively. Then

$$\bar{\Lambda} = \bar{\Lambda}' \cup \bar{\Lambda}''.$$

We shall write the set $\bar{\Lambda}'$ as

$$\bar{\lambda}'_1 > \bar{\lambda}'_2 > \ldots > \bar{\lambda}'_w,$$

while the set $\bar{\Lambda}''$ will be written as

$$\bar{\lambda}_1 > \bar{\lambda}''_1 > \bar{\lambda}''_2 > \ldots > \bar{\lambda}''_{2v} > \bar{\lambda}_{\bar{n}}.$$

Here

$$w + 2v + 2 = m,$$

with w or v possibly being zero. We denote the set of stationary values corresponding to $\bar{\Lambda}'$ as $\{\bar{\mu}'_i\}_{i=1}^{w}$, while the set of stationary values corresponding to $\bar{\Lambda}''$ is written as

$$\{\bar{\mu}_1\} \cup \{\bar{\mu}_m\} \cup \{\bigcup_{i=1}^{2v} \bar{\mu}''_i\}.$$

It will be convenient to define the following closed intervals:

$$\begin{aligned} I_1 &= [\bar{\mu}_1, \infty). \\ I_m &= (-\infty, \bar{\mu}_m]. \\ I''_i &= [\bar{\mu}''_{2i}, \bar{\mu}''_{2i-1}], \ i = 1, 2, \ldots, v. \end{aligned}$$

It is important to note that, in view of (3.15), the intervals defined above are mutually disjoint.

Theorem 3.1 *Let B be an $(n-1) \times (n-1)$ real symmetric matrix, and let A be the perturbation of B given by (3.1). Assume that condition (3.4) holds, and that $\hat{A}$ is of the form (3.5). Let $\bar{\mu}(\cdot)$ be given by (3.8). Then the following hold:*

1. *There exist* $n-2$ *real eigenvalues* $\{\delta_i\}_{i=1}^{n-2}$ *of* A*, including all the eigenvalues of* $\tilde{D}$ *and* $\bar{n}-1$ *eigenvalues of* $\bar{A}$*, which interlace the* $n-1$ *ordered eigenvalues* $\{\gamma_i\}_{i=1}^{n-1}$ *of* B*; that is,*

$$\gamma_1 \geq \delta_1 \geq \gamma_2 \geq \ldots \geq \gamma_{n-2} \geq \delta_{n-2} \geq \gamma_{n-1}. \tag{3.16}$$

2. *The remaining two eigenvalues of* A *(which are eigenvalues of* $\bar{A}$*), say* $\bar{\delta}_a$ *and* $\bar{\delta}_b$*, are real if and only if*

$$t \in \{\bar{\Lambda}'\} \cup \{I_1\} \cup \{I_m\} \cup \{\bigcup_{i=1}^{v} I_i''\}. \tag{3.17}$$

3. *Furthermore,* $\bar{\delta}_a$ *and* $\bar{\delta}_b$ *are real and distinct if and only if* t *is in the interior of one of the* $v+2$ *intervals in (3.17). In this case, the* $\bar{n}+1$ *eigenvalues of* $\bar{A}$ *are real and distinct.*

4. *If (3.17) holds, we have the following relations involving* $\bar{\delta}_a$ *and* $\bar{\delta}_b$*, where we assume* $\bar{\delta}_a \leq \bar{\delta}_b$*:*

   (a) $t > \bar{\mu}_1 \Longrightarrow \bar{\gamma}_1 < \bar{\delta}_a < \bar{\lambda}_1 < \bar{\delta}_b \leq t.$

   (b) $t = \bar{\mu}_1 \Longrightarrow \bar{\gamma}_1 < \bar{\delta}_a = \bar{\lambda}_1 = \bar{\delta}_b \leq t.$

   (c) $t = \bar{\mu}''_{2i-1} \Longrightarrow \bar{\lambda}''_{2i} < \bar{\delta}_a = \bar{\lambda}''_{2i-1} = \bar{\delta}_b.$

   (d) $t \in (\bar{\mu}''_{2i}, \bar{\mu}''_{2i-1}) \Longrightarrow \bar{\lambda}''_{2i} < \bar{\delta}_a < \bar{\lambda}''_{2i-1} < \bar{\delta}_b$
   or $\bar{\delta}_a < \bar{\lambda}''_{2i} < \bar{\delta}_b < \bar{\lambda}''_{2i-1}.$

   (e) $t = \bar{\mu}''_{2i} \Longrightarrow \bar{\lambda}''_{2i} = \bar{\delta}_a = \bar{\lambda}''_{2i} = \bar{\delta}_b < \bar{\lambda}''_{2i-1}.$

   (f) $t = \bar{\mu}'_i \Longrightarrow \bar{\delta}_a = \bar{\mu}'_i = \bar{\delta}_b.$

   (g) $t = \bar{\mu}_m \Longrightarrow t \leq \bar{\delta}_a = \bar{\lambda}_m = \bar{\delta}_b < \bar{\gamma}_{\bar{n}}.$

   (h) $t < \bar{\mu}_m \Longrightarrow t \leq \bar{\delta}_a < \bar{\lambda}_m < \bar{\delta}_b < \bar{\gamma}_{\bar{n}}.$

   Furthermore, in each of the statements (c)–(f), all values on the right-hand-side of the "$\Longrightarrow$*" are contained in a single interval of the form* $(\bar{\gamma}_{j+1}, \bar{\gamma}_j)$.

Proof. Consider the graph of $g_{\bar{\mu}}(\cdot)$, a typical example of which is given in Fig. 2. We see that if $\bar{\gamma}_{i+1} < \bar{\gamma}_i$, then the equation (3.12) has at least one solution $\bar{\delta}_i$ in $(\bar{\gamma}_{i+1}, \bar{\gamma}_i)$, which, in view of Lemma 3.1, is an eigenvalue of $\bar{A}$. Furthermore, since the characteristic equation of $\bar{A}$ is given by

$$\det(\bar{A} - \lambda I) = \prod_{i=1}^{\bar{n}} (\bar{\gamma}_i - \lambda)[t - \lambda + \sum_{i=1}^{\bar{n}} \left(\frac{\bar{\alpha}_i \bar{\beta}_i}{\bar{\gamma}_i - \lambda} \right)] = 0,$$

it follows that if $\bar{\gamma}_i$ has multiplicity k_i as an eigenvalue of $\bar{D}$, then $\bar{\gamma}_i$ is an eigenvalue of $\bar{A}$ with multiplicity $k_i - 1$. Hence

$$\bar{\gamma}_1 \geq \bar{\delta}_1 \geq \bar{\gamma}_2 \geq \ldots \geq \bar{\gamma}_{\bar{n}-1} \geq \bar{\delta}_{\bar{n}-1} \geq \bar{\gamma}_{\bar{n}}, \tag{3.18}$$

where the $\bar{n} - 1$ numbers $\bar{\delta}_i$, are eigenvalues of $\bar{A}$. Part 1 of the theorem now follows readily. Parts 2 and 3 are consequences of Part 4, which follows directly from consideration of the graph of $g_{\bar{\mu}}(\cdot)$, as in Fig. 2. There the relevant values of t are indicated, with the subscripts on t corresponding to (a)–(h) above. (Note that there are two possibilities for (d).) That $\bar{\delta}_b \leq t$ in (a) and (b) follows from the graph and the fact that the trace of $\bar{A}$ is the sum of the eigenvalues of $\bar{A}$, as does the inequality $t \leq \bar{\delta}_a$ occurring in (g) and (h).

□

Figure 2: Values of t yielding real spectrum for A

Remark 3.2 In Theorem 3.1, we can replace $\bar{\mu}(\cdot)$ with any quadratic function of the form

$$\bar{x}^t \bar{D}\bar{x} - \sum_{i=1}^{\bar{n}} \psi_i(\bar{\alpha}_i\bar{\beta}_i)^{1/2}\bar{x}_i,$$

where $\psi_i = \pm 1$, since this change does not alter the Lagrange multipliers or critical values of $\bar{\mu}(\cdot)$ with respect to $S_{\bar{n}}$.

Theorem 3.1 gives a detailed description of the eigenstructure of the perturbation A under assummption (3.4), and in particular, a complete characterization of when the spectrum of A is real. However, in order to apply the result, one requires an orthogonal diagonalization of B, and this may not be readily available. In the following corollary, sufficient conditions for realness of the spectrum of A are given, without reliance on an orthogonal diagonalization, in case A is given by (1.6); that is, when $\alpha = \beta$.

Corollary 3.1 *Let B be an $(n-1)\times(n-1)$ real symmetric matrix, and consider the perturbation of B given by*

$$A = \begin{pmatrix} B & -\alpha \\ \alpha^t & t \end{pmatrix}, \tag{3.19}$$

where α is a real $(n-1)$-vector. Define

$$\mu(x) = x^t Bx - 2\alpha^t x. \tag{3.20}$$

Let

$$\mu_1 = \max\{\mu(x) \ : \ x^t x = 1\} \tag{3.21}$$

and

$$\mu_k = \min\{\mu(x) \ : \ x^t x = 1\}. \tag{3.22}$$

Then either of the conditions

$$t \geq \mu_1 \tag{3.23}$$

or

$$t \leq \mu_k \tag{3.24}$$

are sufficient for the spectrum of A to be real.

Proof. As was noted in Remark 3.1, condition (3.4) holds for the present perturbation. Now observe that $\mu(\cdot)$ and $\bar{\mu}(\cdot)$ have the same secular function and secular antiderivative, where $\bar{\mu}(\cdot)$ is given by (3.8) with $\bar{\alpha}_i = \bar{\beta}_i$ for

$i = 1, 2, \ldots, \bar{n}$. Since the roots of the secular function $f_\mu(\cdot)$ are the critical points of the secular antiderivative $g_\mu(\cdot)$, the discussion of Cases a and b in Section 2.1 tells us that

$$\lambda_1 \geq \bar{\lambda}_1$$

and

$$\lambda_k \leq \bar{\lambda}_m.$$

From Part 2 of Theorem 2.1 we then have

$$\mu_1 \geq \bar{\mu}_1$$

and

$$\mu_k \leq \bar{\mu}_m.$$

(The last two inequalities can also be deduced directly from the definitions of the functions $\mu(\cdot)$ and $\bar{\mu}(\cdot)$.) The result now follows from Theorem 3.1, and in particular, from Part 4, (a), (b), (g) and (h).

$\square$

The values μ_1 and μ_k in the above corollary, which are the optimal objective function values of the trust region problems $P^\mu_{\max}$ and $P^\mu_{\min}$, respectively, may be efficiently determined numerically by the method of Moré and Sorensen [17].

In the next corollary, we give a Gersgorin-like sufficient condition for realness of the spectrum of A given by (3.19). We shall use the notation $\|\cdot\|$ for both the euclidean norm of an $(n-1)$-vector and the spectral norm of an $(n-1) \times (n-1)$ matrix.

Corollary 3.2 *Let B and A be as in Corollary 3.1. Then a sufficient condition for the spectrum of A to be real is*

$$\| B \| + 2 \| \alpha \| \leq |t|. \tag{3.25}$$

Proof. This follows from the fact that (3.25) implies that either (3.23) or (3.24) hold, and Corollary 3.1.

$\square$

We shall conclude this section with another result regarding the perturbation (3.19). This elementary result is independent of Theorem 3.1, and yields further connections between trust region problems and nonsymmetric perturbations.

Theorem 3.2 *Let B, A and $\mu(\cdot)$ be as in Corollary 3.1. Suppose that $\lambda \in \Lambda$; that is, λ is a Lagrange multiplier of $\mu(\cdot)$ with respect to S_{n-1}. Then $x \in S_\mu(\lambda)$ if and only if λ is an eigenvalue of A with $t = \mu(x)$, in which case an associated eigenvector is*

$$\begin{pmatrix} x \\ 1 \end{pmatrix}.$$

Proof. Upon premultiplying the Lagrange equation

$$Bx - \alpha - \lambda x = 0 \tag{3.26}$$

by x^t and using the fact that $x^t x = 1$, we obtain

$$\mu(x) + \alpha^t x - \lambda = 0. \tag{3.27}$$

The theorem now follows from the eigenvector equation

$$\begin{pmatrix} B - \lambda I & -\alpha \\ \alpha^t & t - \lambda \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = 0. \tag{3.28}$$

$\square$

4 Applications

4.1 Exponential Nonnegativity

In the present subsection it will be seen that the main results of Section 3 can be applied to characterize those matrices which are exponentially nonnegative with respect to the n-dimensional ice-cream cone, and to provide a decomposition theorem for such matrices.

Let us denote the *n-dimensional ice-cream cone* by

$$K_n = \{y \in R^n \ : \ \sum_{i=1}^{n-1} y_i^2 \leq y_n^2, \ y_n \geq 0\}.$$

Equivalently,

$$K_n = \{y \in R^n \ : \ y^t Q_n y \leq 0, \ y_n \geq 0\},$$

where $Q_n = \operatorname{diag}(1, 1, \ldots 1, -1)$. We shall denote the matrix exponential by

$$e^{tA} = \sum_{j=0}^{\infty} (tA)^j / j!,$$

and the boundary of K_n by ∂K_n. The following further notation and terminology will be utilized:

$$\begin{aligned} \Pi(K_n) &= \{A : AK_n \subset K_n\}. \\ e(K_n) &= \{A : e^{tA} \subset \Pi(K_n) \ \ \forall t \geq 0\}. \\ e(\partial K_n) &= \{A : e^{tA}(\partial K_n) \subset \partial K_n \ \ \forall t \geq 0\}. \end{aligned}$$

These sets will be referred to as the *K_n-nonnegative matrices*, *exponentially K_n-nonnegative matrices* and *exponentially ∂K_n-invariant matrices*, respectively. It is readily verified that both $\Pi(K_n)$ and $e(\partial K_n)$ are subsets of $e(K_n)$. Although our discussion will be essentially self-contained, the reader is referred to Berman and Plemmons [2] and Berman, Neumann and Stern [1] for general facts concerning these sets of matrices.

Notice that $A \in e(K_n)$ if and only if for any initial point $y_o \in K_n$, the solution $y(t) = e^{tA} y_o$ of the initial value problem

$$\begin{aligned} \frac{d}{dt} y(t) &= Ay(t); \ \ t \geq 0 \\ y(0) &= y_o \end{aligned}$$

satisfies $y(t) \in K_n \ \ \forall t \geq 0$. Similarly, $A \in e(\partial K_n)$ means that $y(t) \in \partial K_n \ \ \forall y_o = y(0) \in \partial K_n$.

We shall require the following lemma of Stern and Wolkowicz [21], in which $e(K_n)$ and $e(\partial K_n)$ are characterized in terms of tangency-like properties of the vector field $\{Ay\}$ relative to the surface

$$\partial K_n = \{y \in R^n \ : \ y^t Q_n y = 0, \ y_n \geq 0\}.$$

Lemma 4.1 *Let A be a real $n \times n$ matrix. Then the following hold:*

1. *A necessary and sufficient condition for $A \in e(K_n)$ is*

$$y^t Q_n A y \leq 0 \ \ \forall y \in \partial K_n. \tag{4.1}$$

2. *A necessary and sufficient condition for $A \in e(\partial K_n)$ is*

$$y^t Q_n A y = 0 \ \ \forall y \in \partial K_n, \tag{4.2}$$

which is in turn equivalent to A being of the form

$$A = \begin{pmatrix} G + aI & g \\ g^t & a \end{pmatrix}, \tag{4.3}$$

for some real $(n-1)$-vector g and real number a, where the $(n-1) \times (n-1)$ matrix G is skew-symmetric.

We next shall use Corollary 3.1 in order to characterize $e(K_n)$ in terms of the maximal critical value of a specific trust region problem, as well as in terms of the realness of the spectrum of a certain matrix.

Suppose that $A \in e(K_n)$, or equivalently, that (4.1) holds. Let us partition A as

$$\begin{pmatrix} A_1 & c \\ d^t & a_{nn} \end{pmatrix}, \tag{4.4}$$

Then condition (4.1) becomes

$$x^t A_1 x + (c^t - d^t)x - a_{nn} \leq 0 \ \ \forall x \in S_{n-1}. \tag{4.5}$$

Let us define

$$B = \frac{A_1 + A_1^t}{2}. \tag{4.6}$$

From (4.5) it follows that (4.1) is equivalent to

$$\mu(x) := x^t Bx - 2\alpha^t x \leq a_{nn} \ \ \forall x \in S_{n-1}, \tag{4.7}$$

where

$$\alpha = \frac{d-c}{2}. \tag{4.8}$$

Defining

$$\mu_1 = \max\{\mu(x) \ : \ x^t x = 1\}$$

as in (3.21), we see that (4.7) becomes

$$\mu_1 \leq a_{nn}. \tag{4.9}$$

Now, in view of Corollary 3.1, (4.9) implies that the spectrum of the matrix

$$A_r = \begin{pmatrix} B & (c-d)/2 \\ (d^t - c^t)/2 & a_{nn} \end{pmatrix} \tag{4.10}$$

is real. We shall call A_r the *regularization* of A.

The preceding discussion is summarized in the following result.

Theorem 4.1 *Let A be a real $n \times n$ matrix. Then the following hold:*

1. *A is exponentially K_n-nonnegative if and only if (4.9) holds.*

2. *A necessary condition for A to be exponentially K_n-nonnegative is that the spectrum of A_r be real.*

Exercise 4.1

$$A = \begin{pmatrix} -1 & 1 & 1 \\ 4 & 2 & 3 \\ 0 & 1 & a_{33} \end{pmatrix}.$$

We wish to determine those values of a_{33} for which $A \in e(K_n)$. The regularization of A becomes

$$A_r = \begin{pmatrix} -1 & 5/2 & -1/2 \\ 5/2 & 2 & -1 \\ 1/2 & 1 & a_{33} \end{pmatrix}.$$

Therefore

$$\mu(x) = -x_1^2 + 2x_2^2 + 5x_1x_2 - x_1 - 2x_2. \tag{4.11}$$

At this point, one could employ the algorithm of Moré and Sorensen [17] in order to compute μ_1. Alternatively, one can find an orthogonal diagonalization of B with MATLAB, and then generate the graph of $g_\mu(\cdot)$. The eigenvalues of B are thusly found to be $\lambda_1 = 3.4155$ and $\lambda_2 = -2.4155$, while an orthogonal matrix which diagonalizes B is

$$P = \begin{pmatrix} .8702 & .4927 \\ -.4927 & .8702 \end{pmatrix}.$$

Then

$$P\alpha = P \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} = \begin{pmatrix} -.9278 \\ -.6239 \end{pmatrix}.$$

It follows that

$$g_\mu(\lambda) = \lambda - \frac{.9278^2}{(-2.4155 - \lambda)} - \frac{.6239^2}{(3.4155 - \lambda)}. \tag{4.12}$$

The graph of $g_\mu(\cdot)$ is shown in Fig. 3.

Figure 3: Graph of (4.12)

Note that the present example is Case a (since no component of $P\alpha$ is zero), and therefore the depicted graph exhibits all the Lagrange multipliers of $\mu(\cdot)$ as critical points. In particular, $\mu_1 = 4.7910$. Hence

$$A \in e(K_n) \Longleftrightarrow a_{33} \geq 4.7910.$$

Furthermore, if a_{33} satisfies this inequality, then the spectrum of A_r is real. Note, however, that this spectrum will also be real in case

$$a_{33} \leq \mu_4 = -4.3408$$

or if

$$\mu_3 = -.6270 \leq a_{33} \leq \mu_2 = 2.3312.$$

Our main objective in the remainder of this subsection is to prove that every exponentially K_n-nonnegative matrix may be (nonuniquely) represented as the sum of a K_n-nonnegative matrix and an exponentially ∂K_n-invariant matrix. Formally, this decomposition result is stated as follows.

Theorem 4.2

$$e(K_n) = \Pi(K_n) + e(\partial K_n). \tag{4.13}$$

In proving this theorem, we will make use of Theorem 4.1. We shall also require the following result, which provides characterizations of $\Pi(K_n)$ and $e(K_n)$ in terms of definiteness conditions. (For a real symmetric matrix C, the notation $C \leq 0$ will indicate that C is negative semidefinite.)

Theorem 4.3 *Let A be a real $n \times n$ matrix.*

1. *Assume that* $\mathrm{rank(A)} > 1$. *Then a necessary and sufficient condition for*

$$A \in \Pi(K_n) \cup \{-\Pi(K_n)\}$$

is the existence of $\mu \geq 0$ such that

$$A^t Q_n A - \mu Q_n \leq 0. \tag{4.14}$$

2. *A necessary and sufficient condition for $A \in e(K_n)$ is the existence of $\gamma \in R$ such that*

$$Q_n A + A^t Q_n - \gamma Q_n \leq 0. \tag{4.15}$$

Part 1 of Theorem 4.3 is due to Loewy and Schneider [13], while Part 2 is due to Stern and Wolkowicz [21].

We now shall prove the decomposition theorem.

Proof of Theorem 4.2. Let $A \in e(K_n)$. By Part 2 of Theorem 4.1, we know that the spectrum of A_r is real, and we can choose $\delta > 0$ such that all eigenvalues of

$$\tilde{A} = A_r + \delta I$$

are positive. Let Γ be an open disk in the open right-half complex plane, centered at $\psi > 0$, such that the entire spectrum of $\tilde{A}$ lies within Γ. Inside Γ, one can express a branch of the function $f(\lambda) = \lambda^{1/2}$ as

$$f(\lambda) = \sum_{i=0}^{\infty} c_i (\lambda - \psi)^i, \tag{4.16}$$

where the coefficients c_i are all real. According to the theory of matrix functions (see e.g. [8]), $\tilde{A}$ has a real square root given by

$$\tilde{A}^{1/2} = \sum_{i=0}^{\infty} c_i (\tilde{A} - \psi I)^i. \tag{4.17}$$

Let us write

$$A = \tilde{A} + C, \tag{4.18}$$

where

$$C = \frac{1}{2} \begin{pmatrix} A_1 - A_1^t & c + d \\ c^t + d^t & 0 \end{pmatrix}. \tag{4.19}$$

Then $C \in e(\partial K_n)$, and in view of Lemma 4.1, it follows that $\tilde{A}$ and A_r are exponentially K_n-nonnegative.

Since $Q_n A_r$ is symmetric, so is $Q_n \tilde{A}$. Then Part 2 of Theorem 4.3 implies the existence of $\tilde{\gamma}$ such that

$$Q_n \tilde{A} - \tilde{\gamma} Q_n \leq 0. \tag{4.20}$$

From the fact that the spectrum of $\tilde{A}^{1/2}$ is real and positive, it follows that

$$(\tilde{A}^{1/2})^t [Q_n \tilde{A} - \tilde{\gamma} Q_n] \tilde{A}^{1/2} \leq 0. \tag{4.21}$$

Now, (4.17) implies that the matrix $Q_n \tilde{A}^{1/2}$ is symmetric, and therefore (4.21) yields

$$Q_n \tilde{A}^2 - \tilde{\gamma} Q_n \tilde{A} \leq 0. \tag{4.22}$$

Again using the symmetry of $Q_n\tilde{A}$, it follows that

$$\tilde{A}^t Q_n \tilde{A} - \tilde{\gamma} Q_n \tilde{A} \leq 0. \tag{4.23}$$

We can assume without loss of generality that $\tilde{\gamma} \geq 0$, since δ can be increased, if necessary. Then upon combining (4.20) and (4.23), we arrive at

$$\tilde{A}^t Q_n \tilde{A} - \tilde{\gamma}^2 Q_n \leq 0. \tag{4.24}$$

Since rank($\tilde{\text{A}}$) = n, Part 1 of Theorem 4.3 implies

$$\tilde{A} \in \Pi(K_n) \cup \{-\Pi(K_n)\}. \tag{4.25}$$

Then (4.25) yields

$$\tilde{A}e^n \in K_n \cup \{-K_n\},$$

where $e^n = (0, 0, ..., 0, 1)^t$. We can assume that δ has been chosen sufficiently large to ensure that

$$\tilde{a}_{nn} = a_{nn} + \delta \geq 0. \tag{4.26}$$

Hence (4.25) and (4.26) imply that $Ae^n \in K_n$. We conclude that

$$\tilde{A} = A - C \in \Pi(K_n). \tag{4.27}$$

Since $C \in e(\partial K_n)$, it follows that

$$A \in \Pi(K_n) + e(\partial K_n). \tag{4.28}$$

This completes the proof.

□

Exercise 4.2 Let

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Here

$$\langle Ay, Q_3 y\rangle = -(y_2 - 1)^2 \leq 0 \ \ \forall y \in K_3,$$

and therefore A is exponentially K_3-nonnegative. Then

$$A_r = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

and following the proof of Theorem 4.2, $\tilde{A} = A_r + \delta I \in \Pi(K_3)$ provided that δ is chosen sufficiently large. Indeed, if we take $\delta = 2$, then

$$\tilde{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix}.$$

It is readily checked that $\langle \tilde{A}y, Q_3\tilde{A}y\rangle = -2(y_2-1)^2 \leq 0$ and $(\tilde{A}y)_3 \geq 0$ for all $y \in K_3$. Hence $\tilde{A} \in \Pi(K_3)$, and therefore $A \in \Pi(K_3) + e(\partial K_3)$.

Remark 4.1 Given an $n \times n$ exponentially K_n-nonnegative matrix in the regularized form A_r, define

$$\delta^* = \min\{\delta \in R : A_r + \delta I \in \Pi(K_n)\}.$$

Let us denote the eigenvalues of A_r by $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$. It is conjectured that

$$\delta^* = -\frac{(\lambda_1 + \lambda_n)}{2}.$$

This is precisely the minimal value of δ which will ensure that the spectral radius of the matrix $A_r + \delta I$ is in its spectrum. Therefore this conjecture relates to the result of Vandergraft in [22] which asserts that a matrix leaves a proper cone invariant only if its spectral radius is an eigenvalue. (The cone K is said to be *proper* provided that it is closed, convex, possesses nonempty interior, and $K \cap \{-K\} = \{0\}$.) Note that δ^* is generally less than the "sufficiently large" value of δ used in the proof of Theorem 4.2 in order to ensure various properties, including the existence of a real square root $\tilde{A}^{1/2} = (A_r + \delta I)^{1/2}$. As an illustration, consider the matrix A_r in Example 4.2. The spectrum of A_r is $\{-1, -1, -1\}$, and therefore $\delta^* = 1$. Then

$$A_r + \delta^* I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \in \Pi(K_3),$$

but this matrix does not possess a real square root.

Remark 4.2 An interesting open problem is to determine whether the decomposition (4.13) holds for any proper cone $K \subset R^n$; that is

$$e(K) = \Pi(K) + e(\partial K). \tag{4.29}$$

It is not difficult to verify (4.29) for the class of ellipsoidal cones; these are the linear homeomorphisms of K_n. Also, results in Schneider and Vidyasagar [18] may be applied in order to verify (4.29) for the class of polyhedral proper cones, and to show that for a general proper cone K we have

$$e(K) = \overline{\Pi(K) + e(\partial K)}, \tag{4.30}$$

where the bar denotes closure. Hence (4.30) implies that conjecture (4.29) is equivalent to closedness of $\Pi(K) + e(\partial K)$. The sets $\Pi(K)$ and $e(\partial K)$ can be shown to be a proper cone and a subspace, respectively, in the space of $n \times n$ matrices. Such a sum is not necessarily closed, and the conjecture therefore remains unsettled.

4.2 Interlacing and Majorization

Given two real n-vectors x and y with component orderings

$$x_1 \geq x_2 \geq \ldots \geq x_n \tag{4.31}$$

and

$$y_1 \geq y_2 \geq \ldots \geq y_n, \tag{4.32}$$

we say that x is *majorized* by y (notationally, $x \prec y$) provided that

$$\begin{aligned} x_1 &\leq y_1, \\ x_1 + x_2 &\leq y_1 + y_2, \\ &\vdots \\ x_1 + x_2 + \ldots + x_{n-1} &\leq y_1 + y_2 + \ldots + y_{n-1}, \\ x_1 + x_2 + \ldots + x_n &= y_1 + y_2 + \ldots + y_n. \end{aligned}$$

The following is a classical theorem of Schur [19].

Theorem 4.4 *Let A be a real symmetric $n \times n$ matrix with diagonal elements*

$$a_{11} \geq a_{22} \geq \ldots \geq a_{nn}$$

and eigenvalues

$$\delta_1 \geq \delta_2 \geq \ldots \geq \delta_n.$$

Then

$$(a_{11}, a_{22}, \ldots, a_{nn}) \prec (\delta_1, \delta_2, \ldots, \delta_n). \tag{4.33}$$

One proof of Schur's theorem appearing in Mirsky [15], and there attributed to Schneider, makes use of the interlacing (1.5) obtained upon writing A in the form (1.4). (See also Theorem 9.B.1 in Marshall and Olkin [14] or Theorem 4.3.26 in Horn and Johnson [11].) We shall next give an analagous "near-majorization" result and proof for (nonsymmetric) matrices of the form (3.1) satisfying condition (3.4), by applying the "near-interlacing" provided by Theorem 3.1. We shall first introduce some further terminology and notation.

Given real n-vectors x and y satisfying the orderings (4.31) and (4.32), we say that x is *near-majorized* by y (notationally, $x \tilde{\prec} y$) provided that

$$\begin{aligned} x_2 &\leq y_2, \\ x_2 + x_3 &\leq y_2 + y_3, \\ &\vdots \\ x_2 + x_3 + \ldots + x_n &\leq y_2 + y_3 + \ldots + y_n, \\ x_1 + x_2 + \ldots + x_n &= y_1 + y_2 + \ldots + y_n. \end{aligned}$$

Note that $x \tilde{\prec} y$ implies $x_1 \geq y_1$.

Theorem 4.5 *Assume that the hypotheses of Theorem 3.1 hold, and consider the matrix $\bar{A}$ given by (3.7).*

1. *Assume that $t \geq \bar{\mu}_1$. Then the spectrum of $\bar{A}$ is real, $t \geq \bar{\gamma}_1$, and*

   $$(t, \bar{\gamma}_1, \bar{\gamma}_2, \ldots, \bar{\gamma}_{\bar{n}}) \tilde{\prec} v,$$

   where v denotes the vector of eigenvalues of $\bar{A}$ listed in non-increasing order.

2. *Assume that $t \leq \bar{\mu}_m$. Then the spectrum of $\bar{A}$ is real, $t \leq \bar{\gamma}_{\bar{n}}$, and*

   $$-(\bar{\gamma}_1, \bar{\gamma}_2, \ldots, \bar{\gamma}_{\bar{n}}, t) \tilde{\prec} -v.$$

Proof. We will only prove Part 1 of the Theorem. The proof of Part 2 is similar and is left to the reader.

In view of Theorem 3.1 (Part 4 (a),(b)), the eigenvalues of $\bar{A}$ are all real and the ordered spectrum of $\bar{A}$ is given by

$$\bar{\delta}_{\bar{n}-1} \leq \bar{\delta}_{\bar{n}-2} \leq \ldots \bar{\delta}_1 \leq \bar{\delta}_a \leq \bar{\delta}_b, \tag{4.34}$$

and we have

$$\bar{\gamma}_{\bar{n}} \leq \bar{\delta}_{\bar{n}-1} \leq \bar{\gamma}_{\bar{n}-1} \leq \ldots \leq \bar{\delta}_1 \leq \bar{\gamma}_1 \leq \bar{\delta}_a \leq \bar{\delta}_b \leq t. \tag{4.35}$$

This yields the system of inequalities

$$\begin{aligned} \bar{\gamma}_1 &\leq \bar{\delta}_a, \\ \bar{\gamma}_1 + \bar{\gamma}_2 &\leq \bar{\delta}_a + \bar{\delta}_1, \\ &\vdots \\ \bar{\gamma}_1 + \bar{\gamma}_2 + \ldots + \bar{\gamma}_{\bar{n}} &\leq \bar{\delta}_a + \bar{\delta}_1 + \ldots + \bar{\delta}_{\bar{n}-1}. \end{aligned}$$

The result now follows from the fact that

$$\begin{aligned} \mathrm{trace}(\bar{\mathrm{A}}) &= t + \bar{\gamma}_1 + \bar{\gamma}_2 + \ldots + \bar{\gamma}_{\bar{n}} \\ &= \bar{\delta}_b + \bar{\delta}_a + \bar{\delta}_1 + \ldots + \bar{\delta}_{\bar{n}-1}. \end{aligned}$$

□

In the following corollary to Theorem 4.5, we obtain a near-majorization result for matrices of the form (1.6).

Corollary 4.1 *Let B, A, $\mu(\cdot)$, μ_1 and μ_k be as in Corollary 3.1, where the diagonal of B is assumed to have the ordering*

$$b_{11} \geq b_{22} \geq \ldots \geq b_{(n-1)(n-1)}. \tag{4.36}$$

In addition, assume that no eigenvector of B is orthogonal to α.

1. *If condition (3.23) holds (that is, $t \geq \mu_1$), then the spectrum of A is real, $t \geq b_{11}$, and*

   $$(t, b_{11}, b_{22}, \ldots, b_{(n-1)(n-1)}) \tilde{\prec} v,$$

   where v is vector of eigenvalues of A listed in non-increasing order.

2. *If condition (3.24) holds (that is, $t \leq \mu_k$), then the spectrum of A is real, $t \leq b_{(n-1)(n-1)}$, and*

   $$-(b_{11}, b_{22}, \ldots, b_{(n-1)(n-1)}, t) \tilde{\prec} -v.$$

Proof. We will only prove Part 1, with the proof of Part 2 being left to the reader.

Since condition (3.6) holds, we have $\hat{A} = \bar{A}$, where $\hat{A}$ and $\bar{A}$ are given by (3.5) and (3.7), respectively. From the proof of Corollary 3.1, we see that the eigenvalues of A are all real, and that the entire ordered spectrum of A is given by (4.34) with $\bar{n} = n - 1$. Upon applying Schur's majorization theorem to B, we obtain

$$\begin{aligned} b_{11} &\leq \bar{\gamma}_1, \\ b_{11} + b_{22} &\leq \bar{\gamma}_1 + \bar{\gamma}_2, \\ &\vdots \\ b_{11} + b_{22} + \ldots + b_{(n-1)(n-1)} &\leq \bar{\gamma}_1 + \bar{\gamma}_2 + \ldots + \bar{\gamma}_{n-1}, \end{aligned}$$

and making use of (4.35) as in the proof of Theorem 4.5, we have that

$$\begin{aligned} \bar{\gamma}_1 &\leq \bar{\delta}_a, \\ \bar{\gamma}_1 + \bar{\gamma}_2 &\leq \bar{\delta}_a + \bar{\delta}_1, \\ &\vdots \\ \bar{\gamma}_1 + \bar{\gamma}_2 + \ldots + \bar{\gamma}_{n-1} &\leq \bar{\delta}_a + \bar{\delta}_1 + \bar{\delta}_2 + \ldots + \bar{\delta}_{n-2}. \end{aligned}$$

The result now follows from the facts that $t \geq \bar{\gamma}_1 \geq b_{11}$ and

$$\begin{aligned} \text{trace(A)} &= t + b_{11} + b_{22} + \ldots + b_{(n-1)(n-1)} \\ &= \bar{\delta}_b + \bar{\delta}_a + \bar{\delta}_1 + \bar{\delta}_2 + \ldots + \bar{\delta}_{n-2}. \end{aligned}$$

□

Notice that the ordering of the diagonal (4.36) can be assumed to hold without loss of generality in Corollary 4.1, since it can always be attained via a permutational similarity.

The following inverse eigenvalue theorem is due to Mirsky [16]; see also Theorem 9.3.B in [14].

Theorem 4.6 *Suppose we are given real numbers $\gamma_1, \gamma_2, \ldots, \gamma_{n-1}$ and $\delta_1, \delta_2, \ldots, \delta_n$ which satisfy the interlacing property (1.5); that is*

$$\delta_1 \geq \gamma_1 \geq \delta_2 \geq \gamma_2 \geq \ldots \geq \gamma_{n-1} \geq \delta_n.$$

Then there exists a real symmetric $n \times n$ matrix of the form

$$A = \begin{pmatrix} D & \eta \\ \eta^t & t \end{pmatrix} \tag{4.37}$$

with $D = \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_{\mathrm{n}-1})$*, such that the spectrum of* A *is* $\{\delta_1, \delta_2, \ldots, \delta_n\}$.

We will now show that an analagous inverse eigenvalue theorem holds when the interlacing (1.5) is replaced by the types of near–interlacing occurring in Theorem 3.1. The proof follows Mirsky's closely, but we include it nevertheless for the sake of completeness. (See also Theorem 7 in [12].)

Theorem 4.7 *Suppose we are given real numbers* $\gamma_1, \gamma_2, \ldots, \gamma_{n-1}$, $\delta_1, \delta_2, \ldots, \delta_{n-2}$ *and* δ_b, δ_a *which satisfy*

$$\gamma_1 \geq \delta_1 \geq \gamma_2 \geq \delta_2 \geq \cdots \geq \delta_{n-2} \geq \gamma_{n-1}. \tag{4.38}$$

Assume that one of the following three cases holds:

1. $\delta_b \geq \delta_a \geq \gamma_1$;
2. $\gamma_{n-1} \geq \delta_b \geq \delta_a$;
3. $\gamma_j \geq \delta_j \geq \delta_b \geq \delta_a \geq \gamma_{j+1}$ *for some* $j,\ 1 \geq j \geq n-2$.

Then there exists a real $n \times n$ *matrix of the form*

$$A = \begin{bmatrix} D & \eta \\ -\eta^t & t \end{bmatrix}$$

such that $D = \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_{\mathrm{n}-1})$ *and such that the spectrum of* A *is* $\{\delta_1, \cdots, \delta_{n-1}\} \cup \{\delta_b, \delta_a\}$.

Proof. The characteristic equation of A is given by

$$\det(\lambda I - A) = \prod_{i=1}^{n-1} (\lambda - \gamma_i)[\lambda - t + \sum_{j=1}^{n-1} \left(\frac{\eta_j^2}{\lambda - \gamma_j} \right)] = 0. \tag{4.39}$$

We need to choose η and t so that the numbers $\delta_i,\ i = 1, \cdots, n$, are the roots of (4.39), where with some abuse of notation, we refer to δ_b, δ_a as δ_{n-1}, δ_n. First suppose that the δ_i are distinct. Let

$$f(\lambda) = \prod_{i=1}^{n} (\lambda - \delta_i), \quad g(\lambda) = \prod_{i=1}^{n-1} (\lambda - \gamma_i).$$

By direct verification, or by Lagrange's interpolation formula, we have

$$\frac{f(\lambda)}{g(\lambda)} = \lambda - (\sum_{i=1}^{n} \delta_i - \sum_{i=1}^{n-1} \gamma_i) + \sum_{k=1}^{n-1} \frac{f(\gamma_k)}{g'(\gamma_k)} \frac{1}{(\lambda - \gamma_k)}). \tag{4.40}$$

Due to the near–interlacing in case (1); that is, when $\delta_b \geq \delta_a \geq \gamma_1$ (or $\delta_{n-1} \geq \delta_n \geq \gamma_1$), we have

$$\begin{aligned} f(\gamma_k) &= \prod_{i=n-1}^{n} (\gamma_k - \delta_i) \prod_{i=1}^{k-1} (\gamma_k - \delta_i) \prod_{i=k}^{n-2} (\gamma_k - \delta_i) \\ &= (-1)^2 \prod_{i=n-1}^{n} |\gamma_k - \delta_i| (-1)^{k-1} \prod_{i=1}^{k-1} |\gamma_k - \delta_i| \prod_{i=k}^{n-2} (\gamma_k - \delta_i) \\ &= (-1)^{k-1} \prod_{i=1}^{n} |\gamma_k - \delta_i| \end{aligned}$$

and

$$g'(\gamma_k) = (-1)^{k-1} \prod_{\substack{i=1 \\ i \neq k}}^{n-1} |\gamma_k - \gamma_i|$$

It follows that

$$\frac{f(\gamma_k)}{g(\gamma_k)} \geq 0, \; k = 1, 2 \cdots, n-1.$$

Similarly, this can be shown to hold in the other two cases as well.

Now choose

$$\eta_k^2 = \frac{f(\gamma_k)}{g'(\gamma_k)}, \; k = 1, 2, \ldots, n,$$

and

$$t = \sum_{i=1}^{n} \delta_i - \sum_{i=1}^{n-1} \gamma_i.$$

Then the eigenvalues of A are the roots of $f(\cdot)$, which are the n values δ_i. (A modification of the proof yields the case of nondistinct δ_i)

□

Remark 4.3 A. Horn [10] proved the following inverse eigenvalue result, which may be viewed as a converse to Theorem 4.4:

If we are given real numbers

$$a_{11} \geq a_{22} \geq \ldots \geq a_{nn}$$

and

$$\delta_1 \geq \delta_2 \geq \ldots \geq \delta_n$$

such that the majorization (4.33) holds, then there exists an $n \times n$ real symmetric matrix A with diagonal elements $a_{11}, a_{22}, \ldots, a_{nn}$ and with eigenvalues $\delta_1, \delta_2, \ldots, \delta_n$.

One proof of Horn's theorem is due to Mirsky [16], and relies on Theorem 4.6. (See also Theorem 9.B.2 in [14].) Hence it seems apppropriate to ask whether one can obtain analagous converses to Theorem 4.5 or Corollary 4.1, by utilizing Theorem 4.7. At the present time, this remains an open problem.

Remark 4.4 In this subsection we have seen that known results for symmetric matrices regarding interlacing and majorization can be extended, under certain conditions, to "near-symmetric" matrices; i.e. matrices of the form (3.19). This was possible because these extensions depended more on the realness of the spectrum than on symmetry per se. Other results in the literature regarding symmetric matrices can also be extended to the near-symmetric case by employing the present work; e.g. results on eigenvalue bounds appearing in Wolkowicz and Styan [24].

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