# A Tight Semidefinite Relaxation of the Cut Polytope

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**Abstract**

We present a tight semidefinite programming (SDP) relaxation for the max-cut problem (MC) which improves on several previous SDP relaxations in the literature. This new SDP relaxation is a tightening of the SDP relaxation recently introduced by the authors, and it inherits all the helpful properties of the latter. We show that it is a strict improvement over the SDP relaxation obtained by adding all the triangle inequalities to the well-known SDP relaxation.

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# 1 Introduction

The max-cut problem (MC) is a combinatorial optimization problem on undirected graphs with weights on the edges. Given such a graph, the problem consists in finding a partition of the set of nodes into two parts so as to maximize the sum of the weights on the edges that have one end in each part of the partition.

It is well known that MC is an NP-hard problem [14], although some special cases can be solved efficiently, for example those where the graph is not contractible to $K_5$, the complete graph on five vertices [3]. In this paper, we consider the general case where the graph is complete. However, all our results hold independent of the type of edge weights. So, in particular, negative or zero edge weights are permitted.

Let the given graph have vertex set $\{1, \ldots, n\}$. We can formulate MC as the optimization of a linear function over the so-called cut polytope $C_n$, which is defined as the convex hull of all $n \times n$ matrices corresponding to possible cuts (a formal definition is given in Section 2.2). Since a complete description of the cut polytope is not available, one approach is to seek good polyhedral relaxations, that is, to approximate the cut polytope by a larger polytope containing it and over which we can optimize in polynomial time using a technique such as linear programming.

One such relaxation of the cut polytope is the metric polytope $M_n$, defined as the set of all matrices satisfying the triangle inequalities. The triangle inequalities model the fact that for any three mutually connected vertices in the graph, it is only possible to cut either zero or two of the edges joining them. In fact, the triangle inequalities are sufficient to describe the cut polytope for graphs with less than five vertices, i.e. $C_n = M_n$ for $n \leq 4$; however, $C_n \subsetneq M_n$ for $n \geq 5$, see for example [4].

Alternatively, we may seek to approximate the cut polytope with non-polyhedral convex sets. Often these sets are defined in terms of symmetric positive semidefinite matrices since we can optimize over the positive semidefinite matrices in polynomial time using semidefinite programming (SDP). This idea was first applied to the stable set problem by Grötschel, Lovász and Schrijver [7]. For MC, the elliptope $\mathcal{E}_n$, which is the set of $n \times n$ correlation matrices, has been used to approximate the cut polytope. Goemans and Williamson [5] use the SDP:

$$\text{(SDP1)} \qquad \nu_1^* \;=\; \begin{array}{ll} \max & \operatorname{trace} QX \\ \text{s.t.} & X \in \mathcal{E}_n \end{array}$$

(the matrix $Q$ is formed from the edge weights; it is formally defined in

Section 2.1) to obtain a cut which is at most 14% above the value of the maximum cut, under the assumption that there are no negative edge weights. The problem SDP1 is also used by Nesterov [18] to provide estimates of the optimal value of MC with constant relative accuracy and with no assumption on the signs of the edge weights. A tighter SDP relaxation of the cut polytope, recently introduced in [2], guarantees an improvement on the optimal value of SDP1 whenever the latter does not yield an optimal solution to MC. This guarantee also requires no assumption on the signs of the edge weights.

A natural idea is to seek to optimize over the set $\mathcal{E}_n \cap M_n$. This was first proposed in [19]. Adding triangle inequalities to SDP1 is considered in e.g. [8, 9, 10]. However, it is not the case that adding a certain subset of the triangle inequalities will improve every instance of max-cut. Furthermore, Karloff [13] has shown that it is impossible to improve the performance guarantee of Goemans and Williamson simply by adding valid linear inequalities to SDP1.

In this paper we present an SDP relaxation for MC which improves on the idea of adding triangle inequalities to SDP1. The feasible set of this tighter SDP relaxation is shown to correspond (under a linear mapping) to a set $F_n$ that is strictly contained in the set $\mathcal{E}_n \cap M_n$ for $n \geq 5$. This SDP relaxation is a tightening of the SDP relaxation presented in [2] but preserves all the helpful properties of the latter.

In the following section we introduce all the notation and definitions we will use. In Section 3 we present the main results from [2] relevant to the introduction of the new tighter relaxation in Section 4. A procedure to test for membership in the set $F_n$ is presented in Section 5. This procedure is used in Section 6 to prove that the tight relaxation is a strict improvement over adding *all* the triangle inequalities to the well-known SDP relaxation. Finally some numerical results are presented in Section 7.

## 2 Notation and Preliminaries

Let $\mathcal{S}^n$ denote the space of $n \times n$ symmetric matrices with the trace inner product $\langle A, B \rangle = \operatorname{trace} AB$. This space has dimension $t(n) := \frac{n(n+1)}{2}$. We will also work in the space $\mathcal{S}^{t(n)+1}$ and for any matrix $Y \in \mathcal{S}^{t(n)+1}$, the $t(n)$ vector $x = Y_{0,1:t(n)}$ denotes the first (zero-th) row of $Y$ after the first element. If $Y$ is a symmetric matrix, $Y \succeq 0$ denotes that it is also positive semidefinite.

We let $e$ denote the vector of ones and $E = ee^T$ the matrix of ones; their dimensions will be clear from the context. We also let $I_n$ denote the identity matrix in $\mathcal{S}^n$ and $e_i$ denote its $i^{\rm th}$ column, so $e_i$ is the $i^{\rm th}$ unit vector. We also define the elementary matrices $E_{ij} = \frac{1}{2}(e_i e_j^T + e_j e_i^T)$. The Hadamard (elementwise) product of matrices $A$ and $B$ is denoted $A \circ B$. We shall also make use of the Frobenius matrix norm:

$$\|A\|_F := \sqrt{\sum_{i,j} A_{ij}^2}.$$

Although it is true that a linear operator on a finite dimensional space can be expressed as a matrix, we use operator notation and operator adjoints because this simplifies notation and improves the clarity of our proofs. We now define the linear operators that we will use.

For $S \in \mathcal{S}^n$, the vector $\mathrm{diag}\,(S) \in \Re^n$ is the **diagonal of** $S$, while the adjoint operator $\mathrm{Diag}\,(v) = \mathrm{diag}^*(v)$ is the **diagonal matrix** with diagonal formed from the vector $v \in \Re^n$. We use both $\mathrm{Diag}\,(v)$ and $\mathrm{Diag}\,v$ if the meaning is clear (similarly for diag and other operators). Also, **the symmetric vectorizing operator** $s = \mathrm{svec}\,(S) \in \Re^{t(n)}$, is formed (columnwise) from $S$ while ignoring the strictly lower triangular part of $S$. Its inverse is the **symmetrizing matrix operator** $S = \mathrm{sMat}\,(s)$. The adjoint of svec is the operator $\mathrm{hMat} = \mathrm{svec}^*$ which forms a symmetric matrix where the off-diagonal terms are halved, i.e. this satisfies

$$\mathrm{svec}\,(S)^T x = \mathrm{trace}\, S\, \mathrm{hMat}\,(x), \qquad \forall S \in \mathcal{S}^n, \quad x \in \Re^{t(n)}.$$

The adjoint of sMat is the operator dsvec which works like svec except that the off-diagonal elements are multiplied by 2, i.e. this satisfies

$$\mathrm{dsvec}\,(S)^T x = \mathrm{trace}\, S\, \mathrm{sMat}\,(x), \qquad \forall S \in \mathcal{S}^n, \quad x \in \Re^{t(n)}.$$

For notational convenience, we define the **symmetrizing diagonal vector** $\mathrm{sdiag}\,(x) := \mathrm{diag}\,(\mathrm{sMat}\,(x))$ and the **vectorizing symmetric vector** $\mathrm{vsMat}\,(x) := \mathrm{vec}\,(\mathrm{sMat}\,(x))$, where vec is the $n^2$-dimensional vector formed from the complete columns of the matrix; the adjoint of vsMat is then given by

$$\mathrm{vsMat}^*(x) = \mathrm{dsvec}\,\left[\frac{1}{2}\left(\mathrm{Mat}\,(x) + \mathrm{Mat}\,(x)^T\right)\right],$$

where Mat is the inverse of vec, i.e. Mat forms an $n \times n$ matrix (columnwise) from an $n^2$-dimensional vector.

In summary,

$$\begin{array}{rcl} \operatorname{diag}^* & = & \operatorname{Diag} \\ \operatorname{svec}^* & = & \operatorname{hMat} \\ \operatorname{svec}^{-1} & = & \operatorname{sMat} \\ \operatorname{dsvec}^* & = & \operatorname{sMat} \\ \operatorname{vsMat}^* & = & \operatorname{dsvec}\left[\frac{1}{2}\left(\operatorname{Mat}(\cdot) + \operatorname{Mat}(\cdot)^T\right)\right] \\ \operatorname{Mat} & = & \operatorname{vec}^{-1}. \end{array}$$

## 2.1 The Max-Cut Problem

Following [17], we can formulate the MC problem as follows. Let the given graph $G$ have vertex set $\{1, \ldots, n\}$ and let it be described by its weighted adjacency matrix $A(G)$. We tacitly assume that the graph in question is complete (if not, missing edges can be given weight 0 to complete the graph). Let $L$ denote the *Laplacian matrix* associated with the graph, hence $L := \operatorname{Diag}(A(G)\,e) - A(G)$. Let the vector $v \in \{+1, -1\}^n$ represent any cut in the graph via the interpretation that the sets $\{i : v_i = +1\}$ and $\{i : v_i = -1\}$ form a partition of the node set of the graph. Then we can formulate MC as:

$$\begin{array}{rll} \mu^* := & \max & \frac{1}{4} v^T L v \\ & \text{s.t.} & v \in \{-1, 1\}^n. \end{array}$$

Using $X := vv^T$, $v^T L v = \operatorname{trace} LX$ and $Q = \frac{1}{4}L$, an equivalent formulation is

$$\begin{array}{rll} \mu^* = & \max & \operatorname{trace} QX \\ & \text{s.t.} & \operatorname{diag}(X) = e \\ & & \operatorname{rank}(X) = 1 \\ & & X \succeq 0. \end{array}$$

Note that $\mu^*$ denotes the optimal value of MC. The various SDP relaxations will have their optimal values denoted by $\nu^*$ with the appropriate subscript.

## 2.2 The Cut Polytope and Relaxations

Our quadratic model for MC with a general homogeneous quadratic objective function is

$$(\text{MC}) \qquad \begin{array}{rcll} \mu^* & = & \max & v^T Q v \\ & & \text{s.t.} & v_i^2 - 1 = 0, \quad i = 1, \ldots, n. \end{array}$$

Note that if the objective function has a linear term, then we can homogenize using an additional variable similarly constrained. Furthermore, we assume $Q \neq 0$ (wlog) in what follows.

Suppose we consider the lifting $X = vv^T$ and $v^T Qv = \operatorname{trace} QX$. We can define the *cut polytope* as the convex hull of the matrices $X$ for all $v \in \{\pm 1\}^n$:

$$C_n := \operatorname{Conv}\{X : X = vv^T, v \in \{\pm 1\}^n\} \subseteq \mathcal{S}^n . \tag{2.1}$$

Furthermore, the matrices $X$ also satisfy the triangle inequalities, which model the fact that for any three vertices of the graph, either two or none of the edges between them are cut. These triangle inequalities define the *metric polytope*:

$$\begin{aligned} M_n := \{X \in \mathcal{S}^n \ : \operatorname{diag}(X) = e, \text{ and} \\ & X_{ij} + X_{ik} + X_{jk} \geq -1, X_{ij} - X_{ik} - X_{jk} \geq -1, \\ & - X_{ij} + X_{ik} - X_{jk} \geq -1, -X_{ij} - X_{ik} + X_{jk} \geq -1, \\ & \forall\, 1 \leq i < j < k \leq n\}. \end{aligned}$$

Finally, they also belong to the *elliptope*:

$$\mathcal{E}_n := \{X \in \mathcal{S}^n \ : \operatorname{diag}(X) = e, X \succeq 0\}. \tag{2.2}$$

Hence, $C_n \subseteq \mathcal{E}_n \cap M_n$ and it is well known that $C_n \neq \mathcal{E}_n \cap M_n$ for $n \geq 5$.

Recall that SDP1 denotes the well-known SDP relaxation for MC that comes from the Lagrangian dual of the Lagrangian dual of MC, see e.g. [2, 21, 20]:

$$\text{(SDP1)} \qquad \begin{array}{rcll} \nu_1^* & = & \max & \operatorname{trace} QX \\ & & \text{s.t.} & \operatorname{diag}(X) = e \\ & & & X \succeq 0. \end{array} \tag{2.3}$$

In other words, SDP1 optimizes over $\mathcal{E}_n$.

# 3 First Strengthening of the SDP1 Relaxation

A stronger SDP relaxation, which we shall refer to as SDP2, was introduced by the authors in [2]. We briefly outline here the derivation of SDP2 and the theoretical results that are relevant for this paper.

The derivation of SDP2 begins by adding the constraint $X^2 - nX = 0$ to SDP1. This constraint is motivated by the observation that $X^2 = vv^T vv^T$, and $v^T v = n$ for all $v \in \{\pm 1\}^n$. We can simultaneously diagonalize $X$ and $X^2$, therefore the eigenvalues of $X$ must satisfy $\lambda^2 - n\lambda = 0$, which implies the only eigenvalues of $X$ are 0 and $n$. This shows that the constraint $X \succeq 0$

becomes redundant and may be removed. Moreover, since the diagonal constraint implies that the trace of $X$ is $n$, we conclude that $X$ is rank-one. We can also add the redundant constraints $X \circ X = E$ to obtain MC2. Note that this constraint (together with $X \succeq 0$) also implies rank-one, see Theorem 3.2 in [2].

We thus have the problem MC2 equivalent to MC:

$$
\text{(MC2)} \qquad \begin{array}{rcc} \mu^* = & \max & \operatorname{trace} QX \\ & \text{s.t.} & \operatorname{diag}(X) = e \\ & & X \circ X = E \\ & & X^2 - nX = 0. \end{array} \tag{3.1}
$$

Taking the dual of MC2 and then the dual of the dual (see [2] for details) yields the following stronger relaxation, which we call SDP2.

$$
\text{(SDP2)} \qquad \begin{array}{rcc} \nu_2^* = & \max & \operatorname{trace} H_c Y \\ & \text{s.t.} & \mathcal{H}_1^*(Y) = n \\ & & \mathcal{H}_2^*(Y) = E \\ & & \mathcal{H}_3^*(Y) = 0 \\ & & \mathcal{H}_4^*(Y) = 1 \\ & & Y \succeq 0, Y \in \mathcal{S}^{t(n)+1}. \end{array} \tag{3.2}
$$

To define the linear operators $\mathcal{H}_i^*(Y), i = 1, 2, 3, 4$, let us partition $Y$ as

$$
Y = \begin{pmatrix} Y_{00} & x^T \\ x & \bar{Y} \end{pmatrix},
$$

where $\bar{Y} \in \mathcal{S}^{t(n)}$. Then:

$\mathcal{H}_1^*(Y) = 2 \operatorname{svec}(I_n)^T x - \operatorname{trace} \operatorname{Diag}(\operatorname{svec}(I_n))\bar{Y}$,
$\mathcal{H}_2^*(Y) = \operatorname{sMat} \operatorname{diag}(\bar{Y})$,
$\mathcal{H}_3^*(Y) = n \operatorname{sMat}(x) - (\operatorname{Mat} \operatorname{vsMat})\, \bar{Y}\, (\operatorname{Mat} \operatorname{vsMat})^*$,
$\mathcal{H}_4^*(Y) = Y_{00}$.

The constraints $\mathcal{H}_2^*(Y) = E$ and $\mathcal{H}_4^*(Y) = 1$ are equivalent to $\operatorname{diag}(Y) = e$. Also, $\mathcal{H}_1^*(Y)$ is twice the sum of the elements in the first row of $Y$ corresponding to the positions of the diagonal of $\operatorname{sMat}(x)$ minus the sum of the same elements in the diagonal of $\bar{Y}$. The constraint $\mathcal{H}_1^*(Y) = n$ implies that $Y_{0,t(i)} = 1, \forall\, i = 1, \ldots, n$, where $t(i) = \frac{i(i+1)}{2}$, and thus it follows that $\operatorname{diag}(\operatorname{sMat}(x)) = e$.

Furthermore, the constraint $\mathcal{H}_3^*(Y) = 0$ implies that the matrix obtained by applying sMat to the vector $x$ (defined in the partition of $Y$ above) of any $Y$ feasible for SDP2 is positive semidefinite.

**Lemma 3.1** *Suppose that* $Y = \begin{pmatrix} Y_{00} & x^T \\ x & \bar{Y} \end{pmatrix}$ *is feasible for SDP2. Then*

$$\text{sMat}\,(x) \succeq 0.$$

**Proof.** Since $\bar{Y}$ is a principal submatrix of $Y$, $\bar{Y} \succeq 0$ holds. The constraint $\mathcal{H}_3^*(Y) = 0$ gives us that

$$\text{sMat}\,(x) = \frac{1}{n}\,(\text{Mat}\,\text{vsMat}\,)\,\bar{Y}\,(\text{Mat}\,\text{vsMat}\,)^*$$

and thus $\text{sMat}\,(x)$ is a congruence of the positive semidefinite matrix $\bar{Y}$. The result follows. ■

**Corollary 3.1** *If* $Y = \begin{pmatrix} Y_{00} & x^T \\ x & \bar{Y} \end{pmatrix}$ *is feasible for SDP2, then* $\text{sMat}\,(x)$ *is feasible for SDP1.*

■

This corollary and some important properties of the nonlinear constraint $X^2 - nX = 0$ yield the following strengthening theorem [2]:

**Theorem 3.1** *The optimal values satisfy*

$$\nu_2^* \leq \nu_1^* \quad \textit{and} \quad \nu_2^* = \nu_1^* \Rightarrow \nu_2^* = \mu^*. \tag{3.3}$$

■

Our objective in the next section is to further tighten the relaxation SDP2. For that purpose, we conclude this section by mentioning that SDP2 may be equivalently expressed as follows:

$$\text{(SDP2)} \qquad \begin{array}{rll} \nu_2^* = & \max & \text{trace}\, H_c Y \\ & \text{s.t.} & \text{diag}\,(Y) = e \\ & & Y_{0,t(i)} = 1,\, i = 1, \ldots, n \\ & & Y_{0,T(i,j)} = \frac{1}{n} \sum\limits_{k=1}^{n} Y_{T(i,k),T(k,j)}, \quad \forall\, i, j \text{ s.t. } 1 \leq i < j \leq n \\ & & Y \succeq 0, Y \in \mathcal{S}^{t(n)+1}, \end{array} \tag{3.4}$$

where

$$T(i,j) := \begin{cases} t(j-1)+i, \text{ if } i \le j \\ t(i-1)+j, \text{ otherwise.} \end{cases}$$

(Recall that $t(i) = \frac{i(i+1)}{2}$, so $T(i,i) = t(i)$.)

We point out that SDP2 has $2t(n)+1$ equality constraints (and the constraints are full rank). The equivalence of these two expressions for SDP2 is shown in [2].

# 4 A Tight Relaxation of the Cut Polytope

To motivate the further tightening of SDP2, let us consider again the rank-one matrices $X = vv^T, v \in \{\pm 1\}^n$. We know that these matrices $X$ have all their entries equal to $\pm 1$. Hence the corresponding matrices $Y$ feasible for SDP2 have all their entries in the first row and column equal to $\pm 1$.

From SDP2, consider the constraints

$$Y_{0,T(i,j)} = \frac{1}{n} \sum_{k=1}^{n} Y_{T(i,k),T(k,j)}, \ \ \forall\, 1 \le i < j \le n,$$

for $Y = \begin{pmatrix} 1 & x^T \\ x & \bar{Y} \end{pmatrix}$ and $x = \text{svec}\,(vv^T)$. The entry $Y_{0,T(i,j)}$ is in the first row of $Y$ and therefore it is equal to 1 in magnitude. The constraint says that it must be equal to the average of $n$ entries in the block $\bar{Y}$. But each of these $n$ entries has magnitude at most 1, because diag $(\bar{Y}) = e$. Hence, for equality to hold, they must all have magnitude equal to 1, and in fact they must all equal $Y_{0,T(i,j)}$.

Let us state this observation in a different way. If $Y$ is rank-one, then the block $\bar{Y} = xx^T$, and therefore $Y_{T(i,k),T(k,j)} = x_{T(i,k)}x_{T(k,j)} = v_i v_k \cdot v_k v_j$. But if $v_k^2 = 1$, then $Y_{T(i,k),T(k,j)} = v_i v_j = X_{ij} = Y_{0,T(i,j)}$.

This discussion leads us to define the relaxation SDP3:

$$\text{(SDP3)} \qquad \begin{array}{rll} \nu_3^* = & \max & \text{trace}\, H_c Z \\ & \text{s.t.} & \text{diag}\,(Z) = e \\ & & Z_{0,t(i)} = 1, i = 1,\ldots,n \\ & & Z_{0,T(i,j)} = Z_{T(i,k),T(k,j)}, \ \forall\, k, \forall\, 1 \le i < j \le n \\ & & Z \succeq 0, Z \in \mathcal{S}^{t(n)+1}. \end{array} \tag{4.1}$$

Note that SDP3 has $(n-1)\cdot t(n-1) + 2n + 1$ equality constraints.

Define

$$F_n := \{X \in \mathcal{S}^n : X = \mathrm{sMat}\,(Z_{0,1:t(n)}), Z \text{ feasible for SDP3}\}.$$

Since the feasible set of SDP3 is convex and compact, and since $F_n$ is the image of that feasible set under a linear transformation, it follows that $F_n$ is also convex and compact.

First we prove that SDP3 is indeed a relaxation of MC. This is not guaranteed a priori since SDP3 is a strengthening of SDP2.

**Lemma 4.1** *$C_n \subseteq F_n$.*

**Proof.** Consider an extreme point of $C_n$, $X = vv^T, v \in \{\pm 1\}^n$. Let $x = \mathrm{svec}\,(X)$ and $Z = \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T$. We show that $Z$ is feasible for SDP3.

Clearly $Z \succeq 0$ and $Z_{0,0} = 1$. Since $x_{T(i,j)} = v_i v_j$, for $1 \le i \le j \le n$,

$$Z_{T(i,j),T(i,j)} = (x_{T(i,j)})^2 = {v_i}^2 {v_j}^2 = 1.$$

Therefore $\mathrm{diag}\,(Z) = e$. Also, $Z_{0,t(i)} = Z_{0,T(i,i)} = x_{T(i,i)} = {v_i}^2 = 1$. Finally, for $1 \le i < j \le n$,

$$\begin{aligned} Z_{T(i,k),T(k,j)} &= x_{T(i,k)} x_{T(k,j)} \\ &= v_i v_k v_k v_j \\ &= v_i v_j \\ &= x_{T(i,j)} \\ &= Z_{0,T(i,j)}. \end{aligned}$$

Hence, each $X = vv^T, v \in \{\pm 1\}^n$ has a corresponding $Z$ feasible for SDP3, and so $X \in F_n$. Since both $C_n$ and $F_n$ are convex, the result follows. ■

Clearly, every $Z$ feasible for SDP3 is feasible for SDP2. Therefore, by Corollary 3.1 above, we have the inclusion:

**Corollary 4.1** *$F_n \subseteq \mathcal{E}_n$.*

■

Using Lemma 4.1, we observe that $\mu^* \le \nu_3^* \le \nu_2^* \le \nu_1^*$. We now claim that

**Theorem 4.1** $F_n \subseteq M_n$.

**Proof.** Suppose $X \in F_n$, then $X = \mathrm{sMat}\,(Z_{0,1:t(n)})$ for some $Z$ feasible for SDP3. Since $Z_{0,t(i)} = 1\ \forall\, i$, it follows that $\mathrm{diag}\,(X) = e$ holds.

Given $i, j, k$ such that $1 \leq i < j < k \leq n$, let $Z_{i,j,k}$ denote the $4 \times 4$ principal minor of $Z$ corresponding to the indices $0, T(i,j), T(i,k), T(j,k)$. Let $a = X_{ij} = Z_{0,T(i,j)}, b = X_{ik} = Z_{0,T(i,k)}, c = X_{jk} = Z_{0,T(j,k)}$. Then

$$Z_{i,j,k} = \begin{pmatrix} 1 & a & b & c \\ a & 1 & c & b \\ b & c & 1 & a \\ c & b & a & 1 \end{pmatrix},$$

since $\mathrm{diag}\,(Z) = e$ and $Z_{0,T(i,j)} = Z_{T(i,k),T(k,j)}$, $Z_{0,T(i,k)} = Z_{T(i,j),T(j,k)}$ and $Z_{0,T(j,k)} = Z_{T(j,i),T(i,k)}$ all hold for $Z$ feasible for SDP3. Now:

$$\begin{aligned}
Z_{i,j,k} \succeq 0 \quad &\Leftrightarrow \quad \begin{pmatrix} 1 & c & b \\ c & 1 & a \\ b & a & 1 \end{pmatrix} - \begin{pmatrix} a \\ b \\ c \end{pmatrix} \begin{pmatrix} a & b & c \end{pmatrix} \succeq 0 \\
&\Leftrightarrow \quad \begin{pmatrix} 1-a^2 & c-ab & b-ac \\ c-ab & 1-b^2 & a-bc \\ b-ac & a-bc & 1-c^2 \end{pmatrix} \succeq 0 \\
&\Rightarrow \quad e^T \begin{pmatrix} 1-a^2 & c-ab & b-ac \\ c-ab & 1-b^2 & a-bc \\ b-ac & a-bc & 1-c^2 \end{pmatrix} e \geq 0.
\end{aligned}$$

Hence,

$$\begin{aligned}
Z_{i,j,k} \succeq 0 \quad &\Rightarrow \quad 3 - (a+b+c)^2 + 2(a+b+c) \geq 0 \\
&\Leftrightarrow \quad \gamma^2 - 2\gamma - 3 \leq 0, \text{where}\, \gamma := a+b+c \\
&\Leftrightarrow \quad (\gamma - 3)(\gamma + 1) \leq 0 \\
&\Leftrightarrow \quad -1 \leq \gamma \leq 3 \\
&\Rightarrow \quad a+b+c \geq -1.
\end{aligned}$$

Therefore, $X_{ij} + X_{ik} + X_{jk} \geq -1$ holds for $X$.

Because multiplication of row and column $i$ of $Z_{i,j,k}$ by $-1$ will not affect the positive semidefiniteness of $Z_{i,j,k}$, if we multiply the two rows and two columns of $Z_{i,j,k}$ with indices $T(i,k)$ and $T(j,k)$ and apply the same argument to the resulting matrix, we obtain the inequality $X_{ij} - X_{ik} - X_{jk} \geq -1$. Similarly, the inequalities $-X_{ij} + X_{ik} - X_{jk} \geq -1$ and $-X_{ij} - X_{ik} + X_{jk} \geq -1$ also hold. ■

We have thus proved the following:

**Corollary 4.2** $C_n \subseteq F_n \subseteq \mathcal{E}_n \cap M_n$. ■

In Section 6, we will prove that the inclusions are in fact strict for $n \geq 5$. However, because we do not have an explicit description of $F_n$, first we need to address the issue of testing for membership in $F_n$. This is the focus of the next section.

# 5 Testing for Membership in $F_n$

Recall that $F_n = \{X \in \mathcal{S}^n : X = \mathrm{sMat}\,(Z_{0,1:t(n)}),\ Z$ feasible for SDP3$\}$, so $F_n$ is defined as the image of the feasible set of SDP3 under the linear mapping sMat applied to the first row of every feasible matrix in SDP3. Because the feasible set of SDP3 is not polyhedral, it is not clear how to give an explicit description of $F_n$.

However, given $X \in \mathcal{S}^n$, the question of determining whether $X \in F_n$ can be expressed as:

Given $X \in \mathcal{S}^n$ satisfying $\mathrm{diag}\,(X) = e$, does there exist a matrix $Z$ feasible for SDP3 such that $\mathrm{sMat}\,(Z_{0,1:t(n)}) = X$?

In this question, only a subset of the elements of $Z$ are specified, namely the elements of $Z_{0,1:t(n)}$ and $Z_{1:t(n),0}$ plus those fixed by the constraints of SDP3. The remaining elements are considered "free" and we ask whether it is possible to complete them in such a way that the resulting matrix $Z$ is positive semidefinite. This problem is an instance of the positive semidefinite matrix completion problem, which has been extensively studied (see e.g. [6, 15, 11]).

We can associate with the partial matrix $Z$ a finite undirected graph $G_Z = (V_Z, E_Z)$ as follows: let the vertex set be $V_Z := \{0, 1, \ldots, t(n)\}$ and let the edge set $E_Z$ contain the edge $(i,j)$ iff the entry $Z_{i,j}$ is fixed. Then $G_Z$ is said to be *chordal* if every cycle of length $\geq 4$ has a chord, i.e. an edge between two non-consecutive vertices. Grone, Johnson, Sá and Wolkowicz [6] showed that if the diagonal entries of $Z$ are specified and the principal minors composed of fixed entries are all nonnegative, then, if the graph $G_Z$ is chordal, a positive semidefinite completion necessarily exists. In our case, however, it is easy to see that the graph $G_Z$ is not chordal for $n \geq 4$. It suffices to consider the cycle of length 4 depicted in Figure 1; since $(T(i,j), T(k,l)) \notin E_Z$ and $(T(i,k), T(j,l)) \notin E_Z$, we see that the cycle has no chords. So we must follow a different approach.

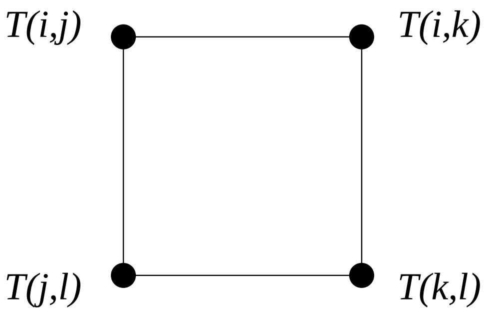


Figure 1: A cycle of length 4 with no chord in the graph $G_Z$ of $Z$.

Johnson, Kroschel and Wolkowicz [12] present an interior-point method for finding an approximate completion, if a completion exists. We use this approach to test membership in $F_n$. Specifically, we proceed as follows: Given $X \in \mathcal{S}^n$ with $\operatorname{diag}(X) = e$, let $x = \operatorname{svec}(X)$ and let $A \in \mathcal{S}^{t(n)+1}$ be some matrix which satisfies $\operatorname{sMat}(A_{0,1:t(n)}) = X$ and furthermore satisfies all the constraints of SDP3, except (possibly) for the positive semidefiniteness constraint. Define $H \in \mathcal{S}^{t(n)+1}$ to be the $\{0,1\}$-matrix satisfying $H_{ij} = 0$ if $A_{ij}$ is "free", and $H_{ij} = 1$ otherwise.

For example, if $X = (X_{ij})$ is $3 \times 3$, one possible choice of $A$ is:

$$A = \begin{pmatrix} 1 & 1 & X_{12} & 1 & X_{13} & X_{23} & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ X_{12} & 0 & 1 & 0 & X_{23} & X_{13} & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ X_{13} & 0 & X_{23} & 0 & 1 & X_{12} & 0 \\ X_{23} & 0 & X_{13} & 0 & X_{12} & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where the "free" entries are filled with zeros. The corresponding matrix $H$ is:

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

To check whether $A$ has a positive semidefinite completion, we consider the problem:

$$c^* = \min \quad \|H \circ (A - P)\|_F^2 \quad \text{s.t.} \quad P \succeq 0 \tag{5.1}$$

and its dual:

$$\begin{array}{rcll} d^* & = & \max & \|H \circ (A-P)\|_F^2 - \operatorname{trace} \Lambda P \\ & & \text{s.t.} & 2\ H \circ H \circ (P - A) = \Lambda \\ & & & \Lambda \succeq 0. \end{array} \tag{5.2}$$

(See [12] for more details.) Clearly if $c^* = 0$, then the corresponding primal optimal solution $P^*$ is an exact positive semidefinite completion of $A$. On the other hand, if we find a pair $(\bar{P}, \bar{\Lambda})$ such that $\|H \circ (A - \bar{P})\|_F^2 - \operatorname{trace} \bar{\Lambda}\bar{P} > 0$, then because $c^* \geq d^*$ (by weak duality), it follows that $c^* > 0$ and hence $A$ has no positive semidefinite completion.

Using this approach, we can find examples which prove that the inclusions in Corollary 4.2 are in fact strict for $n = 5$, and hence for all $n \geq 5$.

# 6 Examples Proving Strict Inclusions

In this section we prove that the inclusions in Corollary 4.2 are strict.

**Example 6.1** *Consider the matrix*

$$X = \begin{pmatrix} 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & 1 & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 1 & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 1 \end{pmatrix}.$$

*It is known that $X \notin C_n$ [16]. Applying the algorithm described in the previous section, we obtain a $16 \times 16$ matrix $P^*$ which is feasible for SDP3 and such that $\operatorname{sMat}(P^*_{0,1:t(n)}) = X$. Hence $X \in F_n$. The matrix $P^*$ is included in Appendix A.*

**Example 6.2** *Consider the matrix*

$$X = \begin{pmatrix} 1 & -0.65 & -0.65 & -0.65 & 0.93 \\ -0.65 & 1 & 0.3 & 0.3 & -0.65 \\ -0.65 & 0.3 & 1 & 0.3 & -0.65 \\ -0.65 & 0.3 & 0.3 & 1 & -0.65 \\ 0.93 & -0.65 & -0.65 & -0.65 & 1 \end{pmatrix}$$

*It is easy to check that $X \in \mathcal{E}_n \cap M_n$. However, using the method described in the previous section, we find feasible matrices $\bar{P}$ and $\bar{\Lambda}$ for which the dual objective value is equal to $2.81\mathrm{e}-4 > 0$. Hence $c^* > 0$ holds and there is no matrix $P$ feasible for SDP3 such that $\mathrm{sMat}\,(P_{0,1:t(n)}) = X$. Hence $X \notin F_n$. The matrices $\bar{P}$ and $\bar{\Lambda}$ are included in Appendix B.*

Hence our final result is:

**Corollary 6.1** *$C_n \subsetneq F_n \subsetneq \mathcal{E}_n \cap M_n$ for $n \geq 5$.*

■

# 7 Numerical Comparison of the Relaxations

The relaxations SDP1, SDP2 and SDP3 were compared for several interesting problems using the software package SDPPACK (version 0.9 Beta) [1]. The results are summarized in Table 1. The value $\rho$ equals the value of the optimal cut divided by the bound, and R.E. denotes the relative error with respect to the optimal cut.

The test problems in Table 1 are as follows:

- The first line of results corresponds to solving the three SDP relaxations for a 5-cycle with unit edge-weights.
- The second line corresponds to the complete graph on 5 vertices with unit edge-weights on all edges except one, which is given weight zero.
- The third line corresponds to the complete graph on 5 vertices with unit edge-weights. In this example, none of the four relaxations attains the MC optimal value, and in fact they are not distinguishable.
- The fourth line corresponds to the graph defined by the weighted adjacency matrix

$$A(G) = \begin{pmatrix} 0 & 1.52 & 1.52 & 1.52 & 0.16 \\ 1.52 & 0 & 1.60 & 1.60 & 1.52 \\ 1.52 & 1.60 & 0 & 1.60 & 1.52 \\ 1.52 & 1.60 & 1.60 & 0 & 1.52 \\ 0.16 & 1.52 & 1.52 & 1.52 & 0 \end{pmatrix}.$$

  This problem is interesting because it shows a significant difference between SDP3 and all the other relaxations; in this case, SDP3 is the only relaxation that attains the MC optimal value.

| $n$ | MC optimal value | SDP1 bound | SDP2 bound | SDP1 plus all triangle inequalities | SDP3 bound | Graph |
|---|---|---|---|---|---|---|
| 5 | 4 | 4.5225<br>$\rho = 0.8845$<br>R.E.: 13.06% | 4.2889<br>$\rho = 0.9326$<br>R.E.: 7.22% | 4.0000<br>$\rho = 1.0000$<br>R.E.: 0% | 4.0000<br>$\rho = 1.0000$<br>R.E.: 0% | 5-cycle with unit edge weights |
| 5 | 6 | 6.2500<br>$\rho = 0.9600$<br>R.E.: 4.17% | 6.1160<br>$\rho = 0.9810$<br>R.E.: 1.93% | 6.0000<br>$\rho = 1.0000$<br>R.E.: 0% | 6.0000<br>$\rho = 1.0000$<br>R.E.: 0% | $K_5 \backslash e$ with unit edge weights |
| 5 | 6 | 6.2500<br>$\rho = 0.9600$<br>R.E.: 4.17% | 6.2500<br>$\rho = 0.9600$<br>R.E.: 4.17% | 6.2500<br>$\rho = 0.9600$<br>R.E.: 4.17% | 6.2500<br>$\rho = 0.9600$<br>R.E.: 4.17% | $K_5$ with unit edge weights |
| 5 | 9.28 | 9.6040<br>$\rho = 0.9663$<br>R.E.: 3.49% | 9.4056<br>$\rho = 0.9866$<br>R.E.: 1.35% | 9.2961<br>$\rho = 0.9983$<br>R.E.: 0.17% | 9.2800<br>$\rho = 1.0000$<br>R.E.: 0% | Given by $A(G)$ below |
| 10 | 12 | 12.5<br>$\rho = 0.9600$<br>R.E.: 4.17% | 12.3781<br>$\rho = 0.9695$<br>R.E.: 3.15% | 12.0000<br>$\rho = 1.0000$<br>R.E.: 0% | 12.0000<br>$\rho = 1.0000$<br>R.E.: 0% | Petersen with unit edge weights |
| 12 | 88 | 90.3919<br>$\rho = 0.9735$<br>R.E.: 2.72% | 89.5733<br>$\rho = 0.9824$<br>R.E.: 1.79% | 88.0029<br>$\rho = 1.0000$<br>R.E.: $3.3e-5$ | 88.0000<br>$\rho = 1.0000$<br>R.E.: $9.9e-7$ | Randomly generated |

Table 1: Numerical results for small test problems

- The last two lines correspond to slightly larger graphs. The graph on 10 vertices is the Petersen graph with unit edge-weights. The graph on 12 vertices is a randomly generated graph (the corresponding matrix $Q$ is included in Appendix C) that gives slightly different results for each relaxation.

In Table 1, a relative error equal to zero means that the relative error was below $10^{-11}$, the value of the smallest default stopping criteria used by SDPpack. Note that the programs SDP2 and SDP3 do not satisfy the Slater constraint qualification (strict feasibility). To avoid numerical errors a projection is done onto the minimal face of the semidefinite cone containing the feasible set. (See [2] for more details.)

We conclude by pointing out that because of the large dimension of the variable matrix $Z$ and because the semidefinite problem has

$$(n-1) \cdot t(n-1) + 2n + 1 = O(n^3)$$

equality constraints, solving the relaxation SDP3 using an interior-point method becomes very time-consuming and requires large amounts of memory, even for moderate values of $n$. It is however important to note that the constraints are very sparse and have a special structure. Current research is being done to exploit this structure via the implementation of a specialized algorithm and thus to make it possible to efficiently solve SDP3 for large instances of MC.

## Acknowledgment

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# A Completed Matrix $P^*$ for Example 6.1

The matrix $P^*$ that follows is the completed positive semidefinite matrix found by the completion approach described in Section 5 and satisfying

$$\mathrm{sMat}\,(P^*_{0,1:t(n)}) = \begin{pmatrix} 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & 1 & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 1 & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 1 \end{pmatrix}.$$

The matrix $P^*$ is formed with the following columns:

```
Columns 1 through 7

 1.0000    1.0000   -0.2500    1.0000   -0.2500   -0.2500    1.0000
 1.0000    1.0000   -0.2500    1.0000   -0.2500   -0.2500    1.0000
-0.2500   -0.2500    1.0000   -0.2500   -0.2500   -0.2500   -0.2500
 1.0000    1.0000   -0.2500    1.0000   -0.2500   -0.2500    1.0000
-0.2500   -0.2500   -0.2500   -0.2500    1.0000   -0.2500   -0.2500
-0.2500   -0.2500   -0.2500   -0.2500   -0.2500    1.0000   -0.2500
 1.0000    1.0000   -0.2500    1.0000   -0.2500   -0.2500    1.0000
-0.2500   -0.2500   -0.2500   -0.2500   -0.2500    0.3750   -0.2500
-0.2500   -0.2500   -0.2500   -0.2500    0.3750   -0.2500   -0.2500
-0.2500   -0.2500    0.3750   -0.2500   -0.2500   -0.2500   -0.2500
 1.0000    1.0000   -0.2500    1.0000   -0.2500   -0.2500    1.0000
-0.2500   -0.2500   -0.2500   -0.2500   -0.2500    0.3750   -0.2500
-0.2500   -0.2500   -0.2500   -0.2500    0.3750   -0.2500   -0.2500
-0.2500   -0.2500    0.3750   -0.2500   -0.2500   -0.2500   -0.2500
-0.2500   -0.2500    0.3750   -0.2500    0.3750    0.3750   -0.2500
 1.0000    1.0000   -0.2500    1.0000   -0.2500   -0.2500    1.0000
```

Columns 8 through 14

| | | | | | | |
|---|---|---|---|---|---|---|
| -0.2500 | -0.2500 | -0.2500 | 1.0000 | -0.2500 | -0.2500 | -0.2500 |
| -0.2500 | -0.2500 | -0.2500 | 1.0000 | -0.2500 | -0.2500 | -0.2500 |
| -0.2500 | -0.2500 | 0.3750 | -0.2500 | -0.2500 | -0.2500 | 0.3750 |
| -0.2500 | -0.2500 | -0.2500 | 1.0000 | -0.2500 | -0.2500 | -0.2500 |
| -0.2500 | 0.3750 | -0.2500 | -0.2500 | -0.2500 | 0.3750 | -0.2500 |
| 0.3750 | -0.2500 | -0.2500 | -0.2500 | 0.3750 | -0.2500 | -0.2500 |
| -0.2500 | -0.2500 | -0.2500 | 1.0000 | -0.2500 | -0.2500 | -0.2500 |
| 1.0000 | -0.2500 | -0.2500 | -0.2500 | -0.2500 | 0.3750 | 0.3750 |
| -0.2500 | 1.0000 | -0.2500 | -0.2500 | 0.3750 | -0.2500 | 0.3750 |
| -0.2500 | -0.2500 | 1.0000 | -0.2500 | 0.3750 | 0.3750 | -0.2500 |
| -0.2500 | -0.2500 | -0.2500 | 1.0000 | -0.2500 | -0.2500 | -0.2500 |
| -0.2500 | 0.3750 | 0.3750 | -0.2500 | 1.0000 | -0.2500 | -0.2500 |
| 0.3750 | -0.2500 | 0.3750 | -0.2500 | -0.2500 | 1.0000 | -0.2500 |
| 0.3750 | 0.3750 | -0.2500 | -0.2500 | -0.2500 | -0.2500 | 1.0000 |
| -0.2500 | -0.2500 | -0.2500 | -0.2500 | -0.2500 | -0.2500 | -0.2500 |
| -0.2500 | -0.2500 | -0.2500 | 1.0000 | -0.2500 | -0.2500 | -0.2500 |

Columns 15 through 16

| | |
|---|---|
| -0.2500 | 1.0000 |
| -0.2500 | 1.0000 |
| 0.3750 | -0.2500 |
| -0.2500 | 1.0000 |
| 0.3750 | -0.2500 |
| 0.3750 | -0.2500 |
| -0.2500 | 1.0000 |
| -0.2500 | -0.2500 |
| -0.2500 | -0.2500 |
| -0.2500 | -0.2500 |
| -0.2500 | 1.0000 |
| -0.2500 | -0.2500 |
| -0.2500 | -0.2500 |
| -0.2500 | -0.2500 |
| 1.0000 | -0.2500 |
| -0.2500 | 1.0000 |

# B Matrices $\bar{P}$ and $\bar{\Lambda}$ for Example 6.2

The matrices $\bar{P}$ and $\bar{\Lambda}$ that follow satisfy

$$||H \circ (A - \bar{P})||_F^2 - \operatorname{trace} \bar{\Lambda}\bar{P} = 2.81e-4.$$

The matrix $\bar{P}$ is formed with the following columns:

Columns 1 through 7

| | | | | | | |
|---|---|---|---|---|---|---|
| 1.0074 | 1.0000 | -0.6475 | 1.0000 | -0.6475 | 0.3026 | 1.0000 |
| 1.0000 | 1.0000 | -0.6427 | 0.9927 | -0.6427 | 0.3004 | 0.9927 |
| -0.6475 | -0.6427 | 1.0020 | -0.6427 | 0.3013 | -0.6487 | -0.6427 |
| 1.0000 | 0.9927 | -0.6427 | 1.0000 | -0.6427 | 0.3004 | 0.9927 |
| -0.6475 | -0.6427 | 0.3013 | -0.6427 | 1.0020 | -0.6487 | -0.6427 |
| 0.3026 | 0.3004 | -0.6487 | 0.3004 | -0.6487 | 1.0026 | 0.3004 |
| 1.0000 | 0.9927 | -0.6427 | 0.9927 | -0.6427 | 0.3004 | 1.0000 |
| -0.6475 | -0.6427 | 0.3013 | -0.6427 | 0.3013 | 0.0257 | -0.6427 |
| 0.3026 | 0.3004 | -0.6487 | 0.3004 | 0.0257 | 0.3001 | 0.3004 |
| 0.3026 | 0.3004 | 0.0257 | 0.3004 | -0.6487 | 0.3001 | 0.3004 |
| 1.0000 | 0.9927 | -0.6427 | 0.9927 | -0.6427 | 0.3004 | 0.9927 |
| 0.9300 | 0.9232 | -0.6500 | 0.9232 | -0.6500 | 0.3804 | 0.9232 |
| -0.6475 | -0.6427 | 0.9307 | -0.6427 | 0.3369 | -0.6487 | -0.6427 |
| -0.6475 | -0.6427 | 0.3369 | -0.6427 | 0.9307 | -0.6487 | -0.6427 |
| -0.6475 | -0.6427 | 0.3369 | -0.6427 | 0.3369 | 0.0257 | -0.6427 |
| 1.0000 | 0.9927 | -0.6427 | 0.9927 | -0.6427 | 0.3004 | 0.9927 |

Columns 8 through 14

| | | | | | | |
|---|---|---|---|---|---|---|
| -0.6475 | 0.3026 | 0.3026 | 1.0000 | 0.9300 | -0.6475 | -0.6475 |
| -0.6427 | 0.3004 | 0.3004 | 0.9927 | 0.9232 | -0.6427 | -0.6427 |
| 0.3013 | -0.6487 | 0.0257 | -0.6427 | -0.6500 | 0.9307 | 0.3369 |
| -0.6427 | 0.3004 | 0.3004 | 0.9927 | 0.9232 | -0.6427 | -0.6427 |
| 0.3013 | 0.0257 | -0.6487 | -0.6427 | -0.6500 | 0.3369 | 0.9307 |
| 0.0257 | 0.3001 | 0.3001 | 0.3004 | 0.3804 | -0.6487 | -0.6487 |
| -0.6427 | 0.3004 | 0.3004 | 0.9927 | 0.9232 | -0.6427 | -0.6427 |
| 1.0020 | -0.6487 | -0.6487 | -0.6427 | -0.6500 | 0.3369 | 0.3369 |
| -0.6487 | 1.0026 | 0.3001 | 0.3004 | 0.3804 | -0.6487 | 0.0257 |
| -0.6487 | 0.3001 | 1.0026 | 0.3004 | 0.3804 | 0.0257 | -0.6487 |
| -0.6427 | 0.3004 | 0.3004 | 1.0000 | 0.9232 | -0.6427 | -0.6427 |
| -0.6500 | 0.3804 | 0.3804 | 0.9232 | 1.0000 | -0.6500 | -0.6500 |
| 0.3369 | -0.6487 | 0.0257 | -0.6427 | -0.6500 | 1.0020 | 0.3013 |
| 0.3369 | 0.0257 | -0.6487 | -0.6427 | -0.6500 | 0.3013 | 1.0020 |
| 0.9307 | -0.6487 | -0.6487 | -0.6427 | -0.6500 | 0.3013 | 0.3013 |
| -0.6427 | 0.3004 | 0.3004 | 0.9927 | 0.9232 | -0.6427 | -0.6427 |

```
Columns 15 through 16

-0.6475    1.0000
-0.6427    0.9927
 0.3369   -0.6427
-0.6427    0.9927
 0.3369   -0.6427
 0.0257    0.3004
-0.6427    0.9927
 0.9307   -0.6427
-0.6487    0.3004
-0.6487    0.3004
-0.6427    0.9927
-0.6500    0.9232
 0.3013   -0.6427
 0.3013   -0.6427
 1.0020   -0.6427
-0.6427    1.0000
```

and the matrix $\bar{\Lambda}$ is formed with the following columns:

```
Columns 1 through 7

 0.0148   -0.0000    0.0050   -0.0000    0.0050    0.0052   -0.0000
-0.0000    0.0000         0         0         0         0         0
 0.0050         0    0.0039         0    0.0025    0.0027         0
-0.0000         0         0    0.0000         0         0         0
 0.0050         0    0.0025         0    0.0039    0.0027         0
 0.0052         0    0.0027         0    0.0027    0.0053         0
-0.0000         0         0         0         0         0    0.0000
 0.0050         0    0.0025         0    0.0025         0         0
 0.0052         0    0.0027         0         0    0.0001         0
 0.0052         0         0         0    0.0027    0.0001         0
-0.0000         0         0         0         0         0         0
-0.0000         0    0.0000         0    0.0000         0         0
 0.0050         0    0.0014         0         0    0.0027         0
 0.0050         0         0         0    0.0014    0.0027         0
 0.0050         0         0         0         0         0         0
-0.0000         0         0         0         0         0         0
```

Columns 8 through 14

| | | | | | | |
|---|---|---|---|---|---|---|
| 0.0050 | 0.0052 | 0.0052 | -0.0000 | -0.0000 | 0.0050 | 0.0050 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.0025 | 0.0027 | 0 | 0 | 0.0000 | 0.0014 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.0025 | 0 | 0.0027 | 0 | 0.0000 | 0 | 0.0014 |
| 0 | 0.0001 | 0.0001 | 0 | 0 | 0.0027 | 0.0027 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.0039 | 0.0027 | 0.0027 | 0 | 0.0000 | 0 | 0 |
| 0.0027 | 0.0053 | 0.0001 | 0 | 0 | 0.0027 | 0 |
| 0.0027 | 0.0001 | 0.0053 | 0 | 0 | 0 | 0.0027 |
| 0 | 0 | 0 | 0.0000 | 0 | 0 | 0 |
| 0.0000 | 0 | 0 | 0 | 0.0000 | 0.0000 | 0.0000 |
| 0 | 0.0027 | 0 | 0 | 0.0000 | 0.0039 | 0.0025 |
| 0 | 0 | 0.0027 | 0 | 0.0000 | 0.0025 | 0.0039 |
| 0.0014 | 0.0027 | 0.0027 | 0 | 0.0000 | 0.0025 | 0.0025 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Columns 15 through 16

| | |
|---|---|
| 0.0050 | -0.0000 |
| 0 | 0 |
| 0 | 0 |
| 0 | 0 |
| 0 | 0 |
| 0 | 0 |
| 0 | 0 |
| 0.0014 | 0 |
| 0.0027 | 0 |
| 0.0027 | 0 |
| 0 | 0 |
| 0.0000 | 0 |
| 0.0025 | 0 |
| 0.0025 | 0 |
| 0.0039 | 0 |
| 0 | 0.0000 |

## C Matrix $Q$ for Test Problem with 12 Vertices

Columns 1 through 7

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | -0.5000 | -0.5000 | 0 | -0.5000 | -1.0000 | 0 |
| -0.5000 | 0 | 0 | -0.5000 | -1.0000 | -0.5000 | -0.5000 |
| -0.5000 | 0 | 0 | 0 | -1.0000 | -1.0000 | -0.5000 |
| 0 | -0.5000 | 0 | 0 | 0 | -0.5000 | 0 |
| -0.5000 | -1.0000 | -1.0000 | 0 | 0 | -0.5000 | -0.5000 |
| -1.0000 | -0.5000 | -1.0000 | -0.5000 | -0.5000 | 0 | -0.5000 |
| 0 | -0.5000 | -0.5000 | 0 | -0.5000 | -0.5000 | 0 |
| -0.5000 | 0 | -1.0000 | 0 | -0.5000 | 0 | 0 |
| -0.5000 | -0.5000 | -1.0000 | -0.5000 | -0.5000 | -0.5000 | 0 |
| -0.5000 | -0.5000 | 0 | -1.0000 | -1.0000 | -0.5000 | -1.0000 |
| -0.5000 | -0.5000 | -0.5000 | -0.5000 | -0.5000 | -0.5000 | -0.5000 |
| -0.5000 | -0.5000 | -0.5000 | -0.5000 | -1.0000 | -0.5000 | -0.5000 |

Columns 8 through 12

| | | | | |
|---|---|---|---|---|
| -0.5000 | -0.5000 | -0.5000 | -0.5000 | -0.5000 |
| 0 | -0.5000 | -0.5000 | -0.5000 | -0.5000 |
| -1.0000 | -1.0000 | 0 | -0.5000 | -0.5000 |
| 0 | -0.5000 | -1.0000 | -0.5000 | -0.5000 |
| -0.5000 | -0.5000 | -1.0000 | -0.5000 | -1.0000 |
| 0 | -0.5000 | -0.5000 | -0.5000 | -0.5000 |
| 0 | 0 | -1.0000 | -0.5000 | -0.5000 |
| 0 | 0 | -1.0000 | -1.0000 | -0.5000 |
| 0 | 0 | -0.5000 | -0.5000 | -0.5000 |
| -1.0000 | -0.5000 | 0 | -1.0000 | -0.5000 |
| -1.0000 | -0.5000 | -1.0000 | 0 | -1.0000 |
| -0.5000 | -0.5000 | -0.5000 | -1.0000 | 0 |