test

TRUST REGION SUBPROBLEMS

$$B = B^t \ \text{ real } (n-1) \times (n-1)$$

eigenvalues: $\gamma_1 \geq \cdots \geq \gamma_{n-1}$

$$v(x) = x^t B x$$
$$\text{subject to } x^t x - 1 = 0$$

Lagrangian: $L(x, \lambda) = v(x) - \lambda(x^t x - 1)$

Lagrange multipliers, stationary points $\iff$ eigenvalues, eigenvectors

$Bx - \lambda x = 0 \qquad\qquad Bu_i = \lambda_i u_i$

For example

$$\lambda_1 = max_{x^t x = 1} v(x) = v(u_1)$$

$$\lambda_{n-1} = min_{x^t x = 1} v(x) = v(u_{n-1})$$

(no convexity required)

nonsymmetric border perturbations:

$$A = \begin{bmatrix} B & -\eta \\ \eta^t & t \end{bmatrix}$$

eigenvalues, eigenvectors

direct connections with perturbed quadratic minimization

trust region problems

$$\min_{x^t x = 1} (\text{max})\ \mu(x) = x^t B x - 2\eta^t x$$

Lagrange multipliers, stationary points

TRUST REGION PROBLEMS (known results)

Levenberg 1944, Marquadt 1963 (Least Squares)

Forsythe and Golub, 1965

Goldfeld, Quandt, and Trotter, 1966

Gander, 1981

Sorensen, 1982

More and Sorensen, 1983

Gander, Golub, and Von Matt, 1989

$$\min_{x^t x=1} (\max) \quad x^t B x - 2\eta^t x$$

Lagrangian $\quad L(x,\lambda) = x^t Bx - 2\eta^t x - \lambda(x^t x - 1)$

$$(B - \lambda I)x - \eta = 0$$

$\lambda \in \Lambda$ Lagrange multipliers
$x \in S_\mu(\lambda)$ stationary points

1. $x \in R^{n-1}$, with $x^t x = 1$, is a minimum (maximum) point of $\mu(\cdot)$ over S_{n-1} if and only if there exists a scalar λ such that x and λ together satisfy the Lagrange equation (stationarity condition), with the matrix $B-\lambda I$ being positive (negative) semidefinite. (no gap in nec & suff)

2. The set Λ of Lagrange multipliers of $\mu(\cdot)$ with respect to S_{n-1} is finite. Let Λ be given by

$$\lambda_1 > \lambda_2 > \ldots > \lambda_k,$$

and let $x^{\lambda_i} \in S_\mu(\lambda_i), \quad i = 1, 2, \ldots, k$. Then

$$\mu(x^{\lambda_1}) > \mu(x^{\lambda_2}) > \ldots > \mu(x^{\lambda_k}).$$

In particular, the minimum (maximum) of $\mu(\cdot)$ over S_{n-1} is attained at any stationary point associated with λ_k (λ_1).

SOLUTION TECHNIQUES FOR TRUST REGION PROBLEMS

easy case: no eigenvector of B is orthogonal to η; $B - \lambda I$ is invertible for all $\lambda \in \Lambda$ and $x^\lambda = (B - \lambda I)^{-1}\eta$.

implicit secular equation:

$$1 - \eta^t (B - \lambda I)^{-2} \eta = 0$$

denoting the spectrum of B by $\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_{n-1}$, let P be an orthogonal matrix such that

$$P^t B P = D = \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_{\mathrm{n}-1}),$$

the diagonal matrix with diagonal elements $\gamma_1, \gamma_2, \ldots, \gamma_{n-1}$. Then the Lagrange stationarity condition becomes

$$(D - \lambda I)\widehat{x} = \widehat{\eta},$$

where

$$\widehat{x} = P^t x, \quad \widehat{\eta} = P^t \eta.$$

The set of Lagrange multipliers Λ is not changed by this transformation of the Lagrange equation, and for every $x \in R^{n-1}$ we have

$$\widehat{\mu}(\widehat{x}) := \widehat{x}^t D\widehat{x} - 2\widehat{\eta}^t\widehat{x} = \mu(x).$$

explicit secular equation

$$f_\mu(\lambda) := 1 - \sum_{i=1}^{n-1} \left(\frac{\widehat{\eta}_i}{\gamma_i - \lambda}\right)^2 = 0.$$

SYMMETRIC BORDER PERTURBATIONS
(well-known properties)

the eigenvalues $\delta_1 \geq \delta_2 \ldots \geq \delta_n$ of the $n \times n$ matrix

$$A = \begin{pmatrix} B & \eta \\ \eta^t & t \end{pmatrix}$$

interlace the eigenvalues of B

$$\delta_1 \geq \gamma_1 \geq \delta_2 \geq \gamma_2 \geq \cdots \geq \gamma_{n-1} \geq \delta_n$$

introduce SECULAR ANTIDERIVATIVE

$$g_\mu(\lambda) = \lambda - \sum_{i=1}^{n-1} \left(\frac{\widehat{\eta}_i^2}{\gamma_i - \lambda} \right),$$

$$g'_\mu(\lambda) = f_\mu(\lambda)$$

connections between trust region problems and perturbation theory

LEMMA Assume easy case and let $\lambda \in \Lambda$. Then the secular antiderivative's values on the Lagrange multiplier set Λ are precisely the values of the quadratic function μ on the corresponding set of stationary points.

$$g_\mu(\lambda) = \mu(x^\lambda),$$

where $x^\lambda = (B - \lambda I)^{-1}\eta$.

GRAPH of secular antiderivative $g_\mu(\cdot)$ in easy case.

$$g_\mu(\lambda) = \lambda - \sum_{i=1}^{n-1} \left(\frac{\widehat{\eta}_i^2}{\gamma_i - \lambda} \right),$$

$g_\mu(\cdot)$ possesses a singularity at each eigenvalue $\gamma_i,\ i = 1, 2, \ldots, n-1$.

$g_\mu(\lambda) \to \infty$ as $\lambda \downarrow \gamma_i$, while $g_\mu(\lambda) \to -\infty$ as $\lambda \uparrow \gamma_i$, for each $i = 1, 2, \ldots, n-1$.

Let $\gamma_{i+1} < \gamma_i$. at least one root of $g_\mu(\cdot)$ in (γ_{i+1}, γ_i); $g_\mu'''(\lambda) < 0$, and consequently $g_\mu''(\lambda)$ is monotone decreasing in this interval.

$$g_\mu(\lambda) = \lambda - \sum_{i=1}^{n-1} \left(\frac{\hat{\eta}_i^2}{\gamma_i - \lambda} \right),$$

$g_\mu(\cdot)$ has at most one point of inflection on (γ_{i+1}, γ_i), which is possibly also a critical point. Should there be a point of inflection in (γ_{i+1}, γ_i), then on that interval $g_\mu(\cdot)$ is strictly convex to the left of this point, and strictly concave to the right of it. Hence $g_\mu(\cdot)$ has either zero, one, or two critical points on (γ_{i+1}, γ_i), with the possibility of only one critical point being accounted for by the existence of an inflection which is also critical.

$g_\mu(\cdot)$ is strictly convex on the semi-infinite interval (γ_1, ∞), while we have strict concavity on the other semi-infinite interval, namely $(-\infty, \gamma_{n-1})$.

$$g_\mu(\lambda) = \lambda - \sum_{i=1}^{n-1} \left(\frac{\hat{\eta}_i^2}{\gamma_i - \lambda} \right)$$

unique critical point, namely λ_1, on (γ_1, ∞)

unique critical point, namely λ_k, on $(-\infty, \gamma_{n-1})$.

set of critical points of the secular antiderivative function $g_\mu(\cdot)$ is Λ.

$$\mu_1 = g_\mu(\lambda_1) > \mu_2 = g_\mu(\lambda_2) > \ldots > \mu_k = g_\mu(\lambda_k),$$

$$\mu_i = \mu(x^{\lambda_i}), \; i = 1, 2, \ldots, k$$

for the *stationary values* of $\mu(\cdot)$ on S_{n-1}.

MAIN RESULTS - preliminary notation

$$A = \begin{pmatrix} B & -\alpha \\ \beta^t & t \end{pmatrix},$$

P orthogonal matrix which diagonalizes B

$$\widehat{P} = \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\widehat{A} := \widehat{P}^t A \widehat{P} = \begin{pmatrix} D & -\widehat{\alpha} \\ \widehat{\beta}^t & t \end{pmatrix},$$

where

$$\widehat{\alpha} = P^t \alpha, \quad \widehat{\beta} = P^t \beta.$$

assumption

$$\widehat{\alpha}_i \widehat{\beta}_i \geq 0, \ \forall i = 1, 2, \ldots, n-1.$$

without loss of generality assume that

$$\hat{A} = \begin{pmatrix} \tilde{D} & 0 & -\tilde{\alpha} \\ 0 & \bar{D} & -\bar{\alpha} \\ \tilde{\beta}^t & \bar{\beta}^t & t \end{pmatrix},$$

where

$$\begin{aligned} \tilde{D} &= \mathrm{diag}(\tilde{\gamma}_1, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_{\tilde{\mathrm{n}}}), \\ \bar{D} &= \mathrm{diag}(\bar{\gamma}_1, \bar{\gamma}_2, \ldots, \bar{\gamma}_{\bar{\mathrm{n}}}), \\ \tilde{\alpha}_i \tilde{\beta}_i &= 0,\ \forall i = 1, 2, \ldots, \tilde{n}, \\ \bar{\alpha}_i \bar{\beta}_i &> 0,\ \forall i = 1, 2, \ldots, \bar{n}, \end{aligned}$$

and

$$\tilde{n} + \bar{n} = n - 1.$$

and the ordering

$$\bar{\gamma}_1 \geq \bar{\gamma}_2 \geq \cdots \geq \bar{\gamma}_{\bar{n}}$$

(‘easy case’) submatrix of $\hat{A}$ given by

$$\bar{A} = \begin{pmatrix} \bar{D} & -\bar{\alpha} \\ \bar{\beta}^t & t \end{pmatrix}.$$

associated quadratic function

$$\bar{\mu}(\bar{x}) = \bar{x}^t \bar{D} \bar{x} - 2 \sum_{i=1}^{\bar{n}} (\bar{\alpha}_i \bar{\beta}_i)^{1/2} \bar{x}_i.$$

associated secular antiderivative function

$$g_{\overline{\mu}}(\lambda) = \lambda - \sum_{i=1}^{\bar{n}} \left(\frac{\bar{\alpha}_i \bar{\beta}_i}{\bar{\gamma}_i - \lambda} \right).$$

characteristic polynomial of A is

$$p(\lambda) = \bar{p}(\lambda) \prod_{i=1}^{\tilde{n}} (\tilde{\gamma}_i - \lambda),$$

where

$$\bar{p}(\lambda) = \det(\bar{A} - \lambda \bar{I}).$$

To characterize the spectrum of A, we need only consider the spectrum of $\bar{A}$, ’easy case’.

LINKAGE

LEMMA The distinct real eigenvalues of $\bar{A}$ which differ from the $\bar{n}$ values $\bar{\gamma}_i$ are the solutions of

$$g_{\overline{\mu}}(\lambda) = t.$$

denote the set of Lagrange multipliers of $\bar{\mu}(\cdot)$ with respect to $S_{\bar{n}}$ by $\bar{\Lambda}$ (easy case),

$$\bar{\lambda}_1 > \bar{\lambda}_2 > \ldots > \bar{\lambda}_m.$$

the set of critical points of $g_{\bar{\mu}}(\cdot)$ is $\bar{\Lambda}$

$$\bar{\mu}_1 = g_{\bar{\mu}}(\bar{\lambda}_1) > \bar{\mu}_2 = g_{\bar{\mu}}(\bar{\lambda}_2) > \ldots > \bar{\mu}_m = g_{\bar{\mu}}(\bar{\lambda}_m),$$

stationary values, points of $\bar{\mu}(\cdot)$ on $S_{\bar{n}}$

$$\bar{\mu}_i = \bar{\mu}(\bar{x}^{\bar{\lambda}_i}),\ i = 1, 2, \ldots, m$$

$$\bar{x}^{\bar{\lambda}_i} = (\bar{D} - \bar{\lambda}_i \bar{I})^{-1} \bar{\eta},\ i = 1, 2, \ldots, m,$$

$\bar{\eta}$ is $\bar{n}$-vector with i^{th} component $\sqrt{\bar{\alpha}_i \bar{\beta}_i}$.

Now provide a qualitative description of the eigenstructure of the border perturbed matrix A when $\alpha_i\beta_i \geq 0$.

Realness of the spectrum of A is characterized in terms of the graph of $g_{\bar{\mu}}(\cdot)$, and in particular, in terms of the stationary values of the quadratic function $\bar{\mu}(\cdot)$. Should the spectrum be real, the interlacing of the eigenvalues of A and B is described.

Let $\bar{\lambda} \in \bar{\Lambda}$. $\bar{\lambda}$ is a *type-1 critical point* of $g_{\bar{\mu}}(\cdot)$ if it is a critical point which is also an inflection. Otherwise $\bar{\lambda}$ a *type-2 critical point* of $g_{\bar{\mu}}(\cdot)$.

denote the sets of type-1 and type-2 critical points of $g_{\bar{\mu}}(\cdot)$ by $\bar{\Lambda}'$ and $\bar{\Lambda}''$, respectively.

$$\bar{\Lambda} = \bar{\Lambda}' \cup \bar{\Lambda}''.$$

$\bar{\Lambda}'$ is $\bar{\lambda}'_1 > \bar{\lambda}'_2 > \ldots > \bar{\lambda}'_w,$
$\bar{\Lambda}''$ is $\bar{\lambda}_1 > \bar{\lambda}''_1 > \bar{\lambda}''_2 > \ldots > \bar{\lambda}''_{2v} > \bar{\lambda}_{\bar{n}}$
disjoint closed intervals:

$$I_1 = [\bar{\mu}_1, \infty), \;\; I_m = (-\infty, \bar{\mu}_m], \;\; I''_i = [\bar{\mu}''_{2i}, \bar{\mu}''_{2i-1}]$$

MAIN RESULT

THEOREM The following hold:

1. There exist $n-2$ real eigenvalues $\{\delta_i\}_{i=1}^{n-2}$ of A, including all the eigenvalues of $\tilde{D}$ and $\bar{n}-1$ eigenvalues of $\bar{A}$, which interlace the $n-1$ ordered eigenvalues $\{\gamma_i\}_{i=1}^{n-1}$ of B; that is,

$$\gamma_1 \geq \delta_1 \geq \gamma_2 \geq \ldots \geq \gamma_{n-2} \geq \delta_{n-2} \geq \gamma_{n-1}.$$

2. The remaining two eigenvalues of A (which are eigenvalues of $\bar{A}$), say $\bar{\delta}_a$ and $\bar{\delta}_b$, are real if and only if

$$t \in \{\bar{\Lambda}'\} \cup \{I_1\} \cup \{I_m\} \cup \{\bigcup_{i=1}^{v} I_i''\}.$$

3. Furthermore, $\bar{\delta}_a$ and $\bar{\delta}_b$ are real and distinct if and only if t is in the interior of one of the $v+2$ intervals. In this case, the $\bar{n}+1$ eigenvalues of $\bar{A}$ are real and distinct.

4. In the real case, we have the following relations involving $\bar{\delta}_a$ and $\bar{\delta}_b$, where we assume $\bar{\delta}_a \leq \bar{\delta}_b$:

(a) $t > \bar{\mu}_1 \Longrightarrow \bar{\gamma}_1 < \bar{\delta}_a < \bar{\lambda}_1 < \bar{\delta}_b \leq \bar{\mu}_1$.

(b) $t = \bar{\mu}_1 \Longrightarrow \bar{\gamma}_1 < \bar{\delta}_a = \bar{\lambda}_1 = \bar{\delta}_b \leq \bar{\mu}_1$.

(c) $t = \bar{\mu}''_{2i-1} \Longrightarrow \bar{\lambda}''_{2i} < \bar{\delta}_a = \bar{\lambda}''_{2i-1} = \bar{\delta}_b$.

(d) $t \in (\bar{\mu}''_{2i}, \bar{\mu}''_{2i-1}) \Longrightarrow \bar{\lambda}''_{2i} < \bar{\delta}_b < \bar{\lambda}''_{2i-1} < \bar{\delta}_a$

or $\bar{\delta}_b < \bar{\lambda}''_{2i} < \bar{\delta}_a < \bar{\lambda}''_{2i}$.

(e) $t = \bar{\mu}''_{2i} \Longrightarrow \bar{\lambda}''_{2i} = \bar{\delta}_a = \bar{\lambda}''_{2i} = \bar{\delta}_b < \bar{\lambda}''_{2i-1}$.

(f) $t = \bar{\mu}'_i \Longrightarrow \bar{\delta}_a = \bar{\mu}'_i = \bar{\delta}_b$.

(g) $t = \bar{\mu}_m \Longrightarrow \bar{\mu}_m \leq \bar{\delta}_a = \bar{\lambda}_m = \bar{\delta}_b < \bar{\gamma}_{\bar{n}}$.

(h) $t < \bar{\mu}_m \Longrightarrow \bar{\mu}_m \leq \bar{\delta}_a < \bar{\lambda}_m < \bar{\delta}_b < \bar{\gamma}_{\bar{n}}$.

COROLLARY Consider the perturbation of B given by

$$A = \begin{pmatrix} B & -\alpha \\ \alpha^t & t \end{pmatrix},$$

where α is a real $(n-1)$-vector. Define

$$\mu(x) = x^t B x - 2\alpha^t x.$$

Let

$$\mu_1 = \max\{\mu(x) \ : \ x^t x = 1\}$$

and

$$\mu_k = \min\{\mu(x) \ : \ x^t x = 1\}.$$

Then either of the conditions

$$t \geq \mu_1$$

or

$$t \leq \mu_k$$

are sufficient for spectrum of A real.

COROLLARY Suppose $\alpha = \beta$. A sufficient condition for the spectrum of A to be real is

$$\| B \| + 2 \| \alpha \| \leq |t|.$$

THEOREM Suppose that $\lambda \in \Lambda$; that is, λ is a Lagrange multiplier of $\mu(\cdot)$ with respect to S_{n-1}. Then $x \in S_\mu(\lambda)$ if and only if λ is an eigenvalue of A with $t = \mu(x)$, in which case an associated eigenvector is

$$\begin{pmatrix} x \\ 1 \end{pmatrix}.$$

APPLICATIONS

exponential cone nonnegativity

cone decompositions

interlacing

majorization

inverse eigenvalue problems

eigenvalue bounds